Ordinary and Partial Differential Equations

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CHAPTER 1

First-order differential equations

The Italian physicist Galileo Galilei famously said that the world is written in the language of mathematics. As it stands, this assertion is certainly questionable from a philosophical standpoint; however it is not controversial that the universe is *read* in mathematical language, which is to say, the physical laws governing it are formulated and studied in mathematical terms.

Whilst elementary algebra, in the form of standard equations, is sufficient to analyze most static problems, it is ill-suited to the understanding of dynamical phenomena, involving physical quantities which change over time. For those, the concept of *differentiation*, one of the foundational notions of calculus, naturally enters the picture, as it formalizes mathematically the intuition of *infinitesimal rate of change* of a certain quantity. An overwhelming majority of the laws of nature, such as Newton's law of gravitation or Maxwell's equations for electromagnetism, express the instantaneous rate of change of a given quantity of interest in terms of other variables of the problem, which might be, for instance, the time variable and the physical quantity under investigation itself. We shall see a wealth of incarnations of this general formulation, starting already with §1.1; in mathematical language, laws of this kind are expressed via *differential equations*, the core topic of this course.

1.1. Differential equations and mathematical models

1.1.1. Some introductory examples. As already alluded to in the introduction to this chapter, a differential equation is a mathematical identity relating a certain unknown function to its derivatives of higher order. For the most part of this course, we shall be interested in functions of a single real variable; differential equations involving those are called *ordinary*, as opposed to *partial differential equations*, the subject of the last chapter of this course, where the unknown is a function of several real variables and the equation relates such function to its *partial derivatives* of higher order.

Before giving the abstract definition of an ordinary differential equation, let us have a look at a few motivating examples.

Example 1.1.1. The equation

$$x'(t) = 2x(t) - t$$

involves an unknown function $x(t)$, of the independent variable t, and its derivative $x'(t)$. We shall classify this example as a *first-order differential equation*.

Example 1.1.2. The equation

$$y''(t) + y'(t) - 3y(t) = \cos t$$

features an unknown function $y(t)$ together with its first two derivatives $y'(t)$, $y''(t)$. We shall refer to it as a *second-order differential equation*.

Example 1.1.3. Consider the equation

$$y'(x) = 3x^2 y(x) \ , \tag{1.1.1}$$

which involves an unknown function $y(x)$ and its derivative $y'(x)$. Let us verify that the **one-parameter family**[1] of functions

$$y(x) = Ce^{x^3}\ , \tag{1.1.2}$$

where C is a constant allowed to range over all real numbers, gives an infinite set of *solutions* to the equation, namely of functions verifying the identity in (1.1.1). To this end, let's compute the derivative $y'(x)$ of $y(x)$ using the familiar chain rule:

$$y'(x) = Ce^{x^3}(3x^2)\ .$$

We immediately realize that the latter expression equals precisely $3x^2 y(x)$, as desired.

Notice that, in the last example, we encountered *infinitely many* different functions satisfying the given equation. We shall see that this is a typical feature of differential equations[2]. For the moment, we might ask ourselves: are the functions in (1.1.2) the *only* possible solutions to (1.1.2), or are there any others? over the course of this chapter, we shall learn a variety of methods to explicitly *solve* differential equations such as the one under consideration here, which will enable us to conclude that, in this specific example, there are no solutions other than the ones in (1.1.2).

1.1.2. Mathematical modelling through differential equations. We shall now present two introductory examples of mathematical modeling of a physical phenomenon governed by a law which lends itself to a formulation via a differential equation.

EXAMPLE 1.1.4 (Newton's law of cooling). In thermodynamics, *Newton's law of cooling* describes the time evolution of the temperature of an object in terms of the temperature of the surrounding environment, such as a hot rock immersed in a glass of cold water. If the surrounding medium is substantially larger than the object under scrutiny, it is physically reasonable to assume that the ambient temperature remains constant in time, namely is unaltered by the interaction with the smaller object. Newton's law of cooling then asserts that the rate of change of the temperature of the object is directly proportional to the difference between the temperature of the object and the one of the ambient space.

Let us model this phenomenon mathematically, specifically let us phrase Newton's law in mathematical terms. Let A denote the temperature of the environment. If $T(t)$ indicates the temperature of the object at time t, then its rate of change is expressed, as is well known from earlier Calculus courses, by the derivative $T'(t)$. We may thus formulate Newton's law as the differential equation

$$T'(t) = -k(T(t) - A)$$

for a certain proportionality constant $k > 0$ (which is part of the physical data of the problem). The reason for the sign of the proportionality constant, which is negative (beware the minus sign in front of the k), is of physical nature: it is well known from experimental evidence that the temperature of the object will increase if it is lower than the one of the environment ($T(t) < A$ implies $T'(t) > 0$), and decrease if it is higher ($T(t) > A$ implies $T'(t) < 0$).

EXAMPLE 1.1.5 (Torricelli's law). In fluid dynamics, *Torricelli's law* states that the instantaneous rate of change of the volume of a liquid inside a draining tank is proportional to the square root of the depth of the liquid. Let us model Torricelli's law by means of an ODE, assuming for simplicity that the draining tank has cylindrical shape with cross-sectional area $A > 0$. Let $V(t)$ and $y(t)$ denote, respectively, the volume and the depth of the liquid at time t. Then the law asserts that

$$V'(t) = -k\sqrt{y(t)} \tag{1.1.3}$$

[1] The reason for the terminology is obvious: the given family of functions is described by a single real parameter C.

[2] By way of contrast, usual algebraic equations such as polynomial equations in one variable have at most *finitely many* solutions.

for some positive constant $k > 0$ (the amount of water in the tank is decreasing, thus $V'(t)$ must be negative).

At first sight, the differential relation (1.1.3) doesn't look like the differential equations we have encountered so far, in that there appear to be two unknown functions, namely $V(t)$ and $y(t)$. However, since the shape of the tank is cylindrical, there is a clear additional relation between the volume and the depth of the liquid, which is $V(t) = Ay(t)$. Since A is constant in time, (1.1.3) translates into

$$Ay'(t) = -k\sqrt{y(t)}\ ,$$

which is now a differential equation of the single unknown function $y(t)$.

We now present an example where a differential equation models a problem of geometric nature.

EXAMPLE 1.1.6. Let $g(x)$ be a real-valued function of a real variable. Suppose g satisfies the following geometric condition: for every point (x, y) in the graph[3] of g, the tangent line to the graph of g at (x, y) passes through the point $(-y, x)$. We shall see how to translate this geometric condition on the graph of g into a differential equation which is satisfied, that is, solved by g.

Fix a point (x, y) in the graph of g; this means that $y = g(x)$. We begin by finding the equation for the tangent line to the graph of g at the point $(x, g(x))$, after which we are going to impose that such line passes through $(-y, x) = (-g(x), x)$. By definition, the sought after tangent line contains the point $(x, g(x))$ and has slope given by the derivative $g'(x)$ at the point x: its equation, using new variables s and t to avoid confusion, is thus

$$t(s) = g'(x)(s - x) + g(x)\ , \tag{1.1.4}$$

where we emphasize once again that x is fixed and s is the variable in the equation. If we impose now the condition that $(-g(x), x)$ lies on such tangent line, we obtain from (1.1.4) the relation

$$x = g'(x)(-g(x) - x) + g(x)\ ,$$

which is a *bona fide* ordinary differential equation, of the first order, satisfied by $g(x)$. Solving such an equation allows thus to determine all possible differentiable functions whose graph satisfies the geometric property phrased at the beginning.

1.1.3. A general framework for ordinary differential equations. We are now ready to give the formal definition of ordinary differential equation.

DEFINITION 1.1.7 (Ordinary differential equation). An **ordinary differential equation** (henceforth routinely abbreviated ODE) is an equation of the form

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \tag{1.1.5}$$

where $n \geq 1$ is an integer, F is a real-valued continuous function of $n + 2$ real variables, and $y(x)$ is the *unknown function* of the equation, which appears in it together with its derivatives $y'(x), y''(x), \dots, y^{(n)}(x)$ and with the *independent variable* x.

The integer n is called the **order** of the ODE (1.1.5).

We shall say that (1.1.5) is an n-th order ODE; n corresponds to the highest order derivative appearing in the given ODE.

EXAMPLE 1.1.8. To digest the abstract definition, let us place the examples encountered so far within the general framework described by Definition 1.1.7.

[3]Recall that the graph of a function $h(x)$ is the set of pairs $(x, h(x))$, which are pictorially identified with points in the xy-plane, where x varies over the domain of definition of h.

(1) The equation

$$y'(x) = 2y(x) - x$$

which, upon renaming the unknown function and the independent variable, is precisely the one treated in Example 1.1.1, takes the form (1.1.5) for $n = 1$ and $F(t_1, t_2, t_3) = -t_1 + 2t_2 - t_3$, a real-valued function of three real variables t_1, t_2, t_3. Indeed, the equation $F(x, y(x), y'(x)) = 0$ amounts precisely to

$$0 = -x + 2y(x) - y'(x) \ , \quad \text{that is,} \quad y'(x) = 2y(x) - x \ .$$

Since the integer n is equal to 1 in this case, we have an example of a first-order differential equation.

(2) The equation

$$y''(x) + y'(x) - 3y(x) = \cos x \ ,$$

already discussed in Example 1.1.2, takes the form (1.1.5) for $n = 2$ and $F(t_1, t_2, t_3, t_4) = \cos t_1 + 3t_2 - t_3 - t_4$; to check this, simply plug the variables $(x, y(x), y'(x), y''(x))$ into (t_1, t_2, t_3, t_4), so as to obtain

$$0 = \cos x + 3y(x) - y'(x) - y''(x) \ , \quad \text{that is,} \quad y''(x) + y'(x) - 3y(x) = \cos x \ .$$

As $n = 2$, this is an example of a second-order ODE.

(3) Newton's law of cooling (Example 1.1.4) is expressed by the differential equation

$$y'(x) = -k(y(x) - A)$$

for given constants k and A. It is straightforward to verify, as in the two examples above, that we obtain the equation in the form (1.1.5) for $n = 1$ and $F(t_1, t_2, t_3) = k(t_2 - A) + t_3$; it is a first-order ODE.

(4) Torricelli's law (Example 1.1.5) is expressed by the differential equation

$$Ay'(x) = -k\sqrt{y(x)}$$

for given constants k and A. The equation takes the form (1.1.5) for $n = 1$ and $F(t_1, t_2, t_3) = k\sqrt{t_2} + At_3$; it is a first-order ODE.

We now formalize the rather intuitive notion of solution of an ODE. By an *open interval* in $\mathbb{R}$ we mean any set of real numbers of the form (a, b), thus with the boundary points a and b excluded, where a is a real number or $a = -\infty$ and $b > a$ is a real number, potentially $b = +\infty$.

Definition 1.1.9 (Solution of an ODE). A **solution of an ODE**

$$F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0$$

is a real-valued function $u(x)$ of a single real variable x, defined on some open interval $I \subset \mathbb{R}$, which is n-times continuously differentiable[4] on I and satisfies the equality

$$F(x, u(x), \ldots, u^{(n)}(x)) = 0 \quad \text{for every } x \in I.$$

Example 1.1.10. Consider the function

$$u(x) = \frac{1}{C - x} \ , \tag{1.1.6}$$

where C is an arbitrary real constant. Since

$$u'(x) = \frac{1}{(C - x)^2} \ ,$$

we deduce that $u(x)$ is a solution to the differential equation

$$y'(x) = y^2(x) \ .$$

It is defined over two separate open intervals, namely $(-\infty, C)$ and $(C, +\infty)$. As C varies in $\mathbb{R}$, (1.1.6) describes a one-parameter family of solutions to the given first-order ODE.

[4]That is, it can be differentiated n times, and its n-th order derivative $u^{(n)}(x)$ is continuous.

Example 1.1.11. Let us verify that the function

$$u(x) = xe^{-x}$$

is a solution, defined over the whole real line, of the second-order ODE

$$y''(x) + 2y'(x) + y(x) = 0 \ .$$

We compute first

$$u'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

via the product rule for derivatives, and similarly

$$u''(x) = -e^{-x} - (1-x)e^{-x} = e^{-x}(x-2) \ .$$

Therefore, we obtain

$$u''(x) + 2u'(x) + u(x) = e^{-x}(x-2) + e^{-x}(2-2x) + xe^{-x} = e^{-x}(x-2+2-2x+x) = 0 \ ,$$

which shows that $u(x) = xe^{-x}$ solves the given ODE.

Observe that the function

$$v(x) = e^{-x}$$

is also a solution: indeed, we have $v'(x) = -e^{-x}$ and $v''(x) = e^{-x}$, so that

$$v''(x) + 2v'(x) + v(x) = e^{-x} - 2e^{-x} + e^{-x} = 0 \ ,$$

as desired.

In general, an ordinary differential equation may fail to admit any solution. For instance, the first-order ODE

$$y'^2(x) + y^2(x) = -1$$

does not admit any solution. Indeed, the square of any real number is nonnegative, whence

$$u'^2(x) + u^2(x) \geq 0$$

for any differentiable function $u(x)$.

It may also be the case that an ODE admits just one solution. This happens, for instance, of the second-order ODE

$$y''^2(x) + y^2(x) = 0 \ ,$$

which is only solved by the constant function $u = 0$.

The last two are, however, rather pathological examples; as we will amply discuss in the sequel, it is standard for an n-th order ODE to admit an *n-parameter family* of solutions, namely a collection of solutions which is described by n distinct real parameters.

Example 1.1.12. Let us go back to Example 1.1.11, i.e., to the second-order differential equation

$$y''(x) + 2y'(x) + y(x) = 0 \ . \tag{1.1.7}$$

We have verified that the two functions

$$u_1(x) = e^{-x} \ , \quad u_2(x) = xe^{-x}$$

are solutions to the equation. Linearity of derivatives allows us to deduce that any function of the form

$$u(x) = C_1u_1(x) + C_2u_2(x) \ , \tag{1.1.8}$$

where C_1 and C_2 are distinct real constants, is a solution: indeed, we compute

$$\begin{aligned} u''(x) + 2u'(x) + u(x) &= (C_1u_1(x) + C_2u_2(x))'' + 2(C_1u_1(x) + C_2u_2(x))' + C_1u_1(x) + C_2u_2(x) \\ &= C_1u_1''(x) + C_2u_2''(x) + 2(C_1u_1'(x) + C_2u_2'(x)) + C_1u_1(x) + C_2u_2(x) \ ; \end{aligned}$$

rearranging terms appropriately, we obtain

$$\begin{aligned} u''(x) + 2u'(x) + u(x) &= C_1(u_1''(x) + 2u_1'(x) + u_1(x)) + C_2(u_2''(x) + 2u_2'(x) + u_2(x)) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0 \ , \end{aligned}$$

as claimed.

We have thus found a two-parameter family of solutions to the second-order ODE (1.1.7); we shall develop solving strategies for such kind of equations which will enables us to ascertain that there are no other solutions, so that (1.1.8) completely describes the set of solutions to the given ODE.

1.1.4. Equilibrium solutions. One of the main goals of this course is to learn to analysize properties of solutions to differential equations. The most basic solutions to conceive are constant solutions.

DEFINITION 1.1.13 (Equilibrium solution). A solution $u(x)$ of an ordinary differential equation

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \ ,$$

defined over a certain open interval $I \subset \mathbb{R}$, is called an **equilibrium solution**, or simply an **equilibrium**, if there is a real number C such that $u(x) = C$ for all $x \in I$.

It shall be important, whenever we attempt to study an ODE, to first single out its equilibrium solutions, if there are any. This will often be a basic step before implementing appropriate methods to find all other solutions.

Suppose the constant function $u(x) = C$ is a solution to the ODE

$$F(x, y(x), \dots, y^{(n)}(x)) = 0 \ ;$$

by definition, this means that

$$0 = F(x, u(x), u'(x), \dots, u^{(n)}(x)) = F(x, C, 0, \dots, 0) \ ,$$

the last equality holding since all derivatives of a constant function vanish identically. Therefore, the real number C is a solution to the functional equation

$$F(x, C, 0, \dots, 0) = 0 \ ,$$

meaning that $F(x, C, 0, \dots, 0) = 0$ for all x in the domain of definition of F. Conversely, it is clear that if $C \in \mathbb{R}$ solves the last displayed equation, then the constant function $u(x) = C$ is a solution to the given ODE.

EXAMPLE 1.1.14. Let's determine all equilibrium solutions of the first-order ODE

$$y'(x) = (M - y(x))(y^2(x) - 3y(x) + 2) \ ,$$

where M is a given real number. A constant function $u(x) = C$ solves the equation if and only if

$$0 = u'(x) = (M - C)(C^2 - 3C + 2) = (M - C)(C - 1)(C - 2) \ ,$$

which is an algebraic equation in the variable C with solutions $C = 1$, $C = 2$ and $C = M$. Therefore the equilibrium solutions of the ODE at hand are

$$u(x) = 1 \ , \quad u(x) = 2 \ , \quad u(x) = M \ .$$

EXAMPLE 1.1.15. Consider the second-order ODE

$$y''(x) - y'(x) + 3ky(x) = 0 \ ,$$

where $k \in \mathbb{R}$ is given. Let's determine the equilibrium solutions: a constant function $u(x) = C$ solves the ODE if and only if

$$0 = u''(x) - u'(x) + 3ku(x) = 0 + 0 + 3kC = 3kC \ .$$

We have thus two distinct regimes according to the value of k: if $k \neq 0$, then the last displayed algebraic equation is only solved for $C = 0$, which produces the unique equilibrium solution

$$u(x) = 0 \ .$$

On the other hand, if $k = 0$, then the equation $0 = 3kC$ is always verified, no matter the value of C; in this case, we thus have a one-parameter family of equilibrium solutions to the given ODE,

$$u(x) = C\ ,\quad C \in \mathbb{R}.$$

EXAMPLE 1.1.16. Consider the first-order ODE

$$y'(x) = y(x)\cos x - e^x\ .$$

Let $u(x) = C$ be a constant function: can it be a solution to the given equation? For it to be the case, we must have

$$0 = u'(x) = u(x)\cos x - e^x = C\cos x - e^x\ ,$$

for all real values of x. It is clear that there exists no real number C for which this is verified, since $C\cos x$ is a bounded function, whereas e^x is unbounded. Thus, the given ODE has no equilibrium solutions.

1.1.5. Initial value problems. In applications, differential equations customarily appear in conjunction with *initial conditions*: in the study of the time-evolution a certain physical quantity $y(t)$, we typically know its value y_0 at a given moment in time t_0, and we understand the physical law underlying its evolution, expressed by a differential equation for $y(t)$. Assuming such knowledge, we would like to determine the future evolution of $y(t)$ completely, namely all the values $y(t)$ for all $t > t_0$. A problem of such nature is known as *initial value problem.*

DEFINITION 1.1.17 (Initial value problem). An **initial value problem** (IVP in abridged form) is a pair

$$\begin{cases} F(x, y(x), \dots, y^{(n)}(x)) = 0 \\ y(x_0) = y_0\ ,\ y'(x_0) = y_1\ ,\ \dots\ ,\ y^{(n-1)}(x_0) = y_{n-1} \end{cases} \tag{1.1.9}$$

consisting of an ordinary differential equation

$$F(x, y(x), \dots, y^{(n)}(x)) = 0 \tag{1.1.10}$$

and a set of **initial conditions**

$$y(x_0) = y_0\ ,\ y'(x_0) = y_1\ ,\ \dots\ ,\ y^{(n-1)}(x_0) = y_{n-1}$$

where $x_0, y_0, \dots, y_{n-1}$ are real numbers.

A **solution of the IVP** (1.1.9) is a solution $u(x)$ of the ODE (1.1.10) which is defined on an interval I containing the point x_0, and which satisfies the conditions

$$u(x_0) = y_0\ ,\ u'(x_0) = y_1\ ,\ \dots\ ,\ u^{(n-1)}(x_0) = y_{n-1}\ .$$

EXAMPLE 1.1.18. Let us find a solution to the IVP

$$\begin{cases} y'(x) = y^2(x) \\ y(1) = 1 \end{cases}$$

taking advantage of the family of solutions found in Example 1.1.10. From the latter, we know that each function

$$u(x) = \frac{1}{C - x}\ ,$$

for $C \in R$, is a solution to the ODE in the given initial value problem. We now *impose the initial condition* $u(1) = 1$ prescribed by the IVP, and obtain the algebraic equation

$$1 = u(1) = \frac{1}{C - 1}\ ,$$

from which we readily get $C = 2$. Therefore, the function

$$u(x) = \frac{1}{2 - x}$$

is a solution to the given IVP.

It is now natural to ask: are there any more solutions? Certainly none of the form $1/(C-x)$ for $C \neq 2$, since we obtained $C = 2$ precisely by dictating the initial condition. In principle, there might however be other solutions to the ODE $y'(x) = y^2(x)$ which are not of the form $u(x) = 1/(C - x)$. In later sections of this chapter we will *prove* that there are, as a matter of fact, no other solutions.

EXAMPLE 1.1.19. Leveraging the family of solutions found in Example 1.1.12, let us find a solution to the IVP

$$\begin{cases} y''(x) + 2y'(x) + y(x) = 0 \\ y(0) = 0 \ , \ y'(0) = -1 \end{cases} .$$

A *general solution* to the given DE is, as we already verified,

$$u(x) = C_1 e^{-x} + C_2 x e^{-x} \tag{1.1.11}$$

for real parameters C_1, C_2. We now impose the two initial conditions:

$$0 = u(0) = C_1 \cdot 1 + C_2 \cdot 0 = C_1 \ ,$$

so that we find $C_1 = 0$ and $u(x) = C_2 x e^{-x}$ for some $C_2 \in \mathbb{R}$, which we determine imposing the second initial condition. We compute $u'(x) = C_2(e^{-x} - xe^{-x})$, and thus

$$-1 = u'(0) = C_2(1 - 0) = C_2 \ .$$

Therefore, the unique function $u(x)$ in the family (1.1.11) which solves the original IVP is

$$u(x) = -xe^{-x} \ .$$

A major achievement of the general mathematical theory of ordinary differential equations is that, under rather mild assumptions, initial value problems *always admit a unique solution* defined for all values of the independent variable x which are sufficiently close to the initial value x_0. Thus, while a n-th order ODE usually admits an n-parameter family of solutions, the additional datum of n initial conditions in an IVP forces uniqueness. In this course, we shall not be concerned with the abstract theory of ODEs, and will rather verify the aforementioned uniqueness principle in a wide variety of specific examples.

1.1.6. Ordinary differential equations in normal form. Throughout this course, we shall exclusively deal with ordinary differential equations expressed in **normal form**, namely those expressing the highest-order derivative of the unknown function as a function of all the remaining derivatives: more precisely, an n-th order ODE in normal form appears as

$$y^{(n)}(x) = F(x, y(x), \dots, y^{(n-1)}(x))$$

for a certain continuous real-valued function F of $n + 1$ real variables.

EXAMPLE 1.1.20. The first-order ODE

$$y'(x) = 2x \log y(x)$$

is in normal form, whereas the first-order ODE

$$y'^2(x) + y^2(x) = x^4$$

is not in normal form. Notice that trying to solve the latter for the highest-order derivative $y'(x)$ would produce the ambiguity

$$y'(x) = \pm\sqrt{x^4 - y^2(x)}$$

in the choice of square root.

EXAMPLE 1.1.21. The second-order ODE

$$t^3 x''(t) - t^2 x(t) = t - \sin t$$

is not in normal form, while the second-order ODE

$$x''(t) = -tx'(t) - x(t) + 1$$

is in normal form.

For the sake of brevity, we adopt the following terminological convention.

CONVENTION. From now on, unless explicitly mentioned, a differential equation without further specification is meant to be an ordinary differential equation, and will routinely be abbreviated as DE.

1.2. Integrals as general and particular solutions

We now begin a systematic study of first-order differential equations in autonomous form, that is, equations of the form

$$y'(x) = f(x, y(x))$$

in the unknown $y(x)$, where f is a (given) function of two real-variables. This section is devoted to the analysis of the most elementary instances of such equations, namely the case where the function f only depends on the independent variable x. The resulting form of the equation is thus

$$y'(x) = f(x)\ . \tag{1.2.1}$$

Direct integration yields all solutions to the last-displayed equation. Indeed, here we have a fixed, known continuous function $f(x)$, and we look for all continuously differentiable functions $y(x)$ whose derivative is given by the function f. According to the terminology introduced in Calculus 2, $y(x)$ solves the DE in (1.2.1) if and only if $y(x)$ is an *anti-derivative* of the function $f(x)$. Anti-derivatives are given by *indefinite integrals*, whence $y(x)$ is a solution if and only if

$$y(x) = \int f(x)\ \mathrm{d}x = g(x) + C \tag{1.2.2}$$

where $g(x)$ is a choice of an anti-derivative of $f(x)$, and C is a real constant. What we just described in (1.2.2) is routinely referred to as **a general solution** of the DE in (1.2.1), namely a collection of solutions parametrized, in this case, by the constant C. For each fixed $C \in \mathbb{R}$, we obtain a **particular solution** of the equation in (1.2.1); thus a general solution is a family of particular solutions. In this case, we are dealing with a one-parameter family of solutions, as shall customarily be the case for first-order differential equations.

If a general solution to a given equation comprises *all* possible solutions, then we shall speak of **the general solution** of the equation. In the present case, (1.2.2) provides the general solution to (1.2.1); indeed, if $g(x)$ is a fixed anti-derivative of $f(x)$ and $u(x)$ is any solution to (1.2.1), namely satisfies $u'(x) = f(x)$, then a well known theorem of calculus tells us that u and g must differ by a constant, for they have the same derivative. Hence, $u(x) = g(x) + C$ for some $C \in \mathbb{R}$ and thus u belongs to the family of functions described in (1.2.2).

Recall that an anti-derivative $g(x)$ of $f(x)$ is given by any *definite integral* of the form

$$g(x) = \int_{x_0}^{x} f(t)\ \mathrm{d}t$$

where x_0 is a real number; this is indeed the content of the fundamental theorem of calculus.

Initial conditions enable to specialize a general solution to a particular solution. Suppose given an IVP

$$\begin{cases} y'(x) = f(x) \\ y(x_0) = y_0 \end{cases} \tag{1.2.3}$$

where the differential equation is of the kind we are studying in this section. We known from the discussion above that a solution to the DE in (1.2.3) must take the form

$$y(x) = g(x) + C$$

where g is a fixed anti-derivative of f and C is a real number. Imposing the initial condition $y(x_0) = y_0$ yields

$$y_0 = y(x_0) = g(x_0) + C \ ,$$

from which we derive

$$C = y_0 - g(x_0) \ .$$

Therefore, we have shown that the IVP in (1.2.3) admits a *unique solution*, which is given by

$$y(x) = g(x) + y_0 - g(x_0)$$

for any fixed anti-derivative $g(x)$ of $f(x)$. If, for instance, we choose $g(x)$ to be the definite integral

$$g(x) = \int_{x_0}^{x} f(t) \ \mathrm{d}t \ ,$$

then $g(x_0) = 0$ and thus the unique solution can be expressed as

$$y(x) = y_0 + \int_{x_0}^{x} f(t) \ \mathrm{d}t \ .$$

We summarize the results obtained so far in this section in the following theorem.

THEOREM 1.2.1. *Let $f(x)$ be a continuous function defined on an open interval $I = (a, b) \subset \mathbb{R}$, and consider the first-order differential equation*

$$y'(x) = f(x) \ .$$

Let $g(x)$ be an anti-derivative of $f(x)$ on I. Then, a continuously differentiable function $u(x)$, defined on I, is a solution of the given differential equation if and only if

$$u(x) = g(x) + C$$

for some $C \in \mathbb{R}$.

Furthermore, if $x_0 \in I$ and $y_0 \in \mathbb{R}$, the initial value problem

$$\begin{cases} y'(x) = f(x) \\ y(x_0) = y_0 \end{cases}$$

admits a unique solution $u(x)$ defined on I, which is given by

$$u(x) = y_0 + \int_{x_0}^{x} f(t) \ \mathrm{d}t \ .$$

We now familiarize ourselves with the method by working out a few examples.

EXAMPLE 1.2.2. Consider the IVP

$$\begin{cases} y'(x) = x + 4 \\ y(1) = 3 \end{cases} .$$

We first find the general solution to

$$y'(x) = x + 4$$

by means of indefinite integrals:

$$y(x) = \int x + 4 \ \mathrm{d}x = \frac{x^2}{2} + 4x + C \ , \quad C \in \mathbb{R}.$$

Now, we impose the condition $y(1) = 3$ to find the appropriate value of C; we have

$$3 = y(1) = \frac{1}{2} + 4 + C \ ,$$

from which

$$C = 3 - \frac{1}{2} - 4 = -\frac{3}{2} \, .$$

We conclude that the unique solution to the given IVP is the function

$$y(x) = \frac{x^2}{2} + 4x - \frac{3}{2} \, .$$

EXAMPLE 1.2.3. Consider the IVP

$$\begin{cases} y'(x) = \frac{1}{\sqrt{x+1}} \\ y(0) = 2 \end{cases} \, .$$

The function $f(x) = \frac{1}{\sqrt{x+1}}$ is defined over the open interval $\{x : x + 1 > 0\} = (-1, +\infty)$. The general solution to

$$y'(x) = \frac{1}{\sqrt{x+1}}$$

is given by

$$y(x) = \int \frac{1}{\sqrt{x+1}} \, \mathrm{d}x = 2\sqrt{x+1} + C \, , \quad C \in \mathbb{R}.$$

Plugging the initial condition $y(0) = 2$ yields

$$2 = y(0) = 2 + C \, ,$$

which gives $C = 0$. Thus the unique solution to the given IVP is the function

$$y(x) = 2\sqrt{x+1} \, .$$

CHAPTER 2

CHAPTER 3

3.1. Mechanical vibrations

3.1.1. Free undamped motion. We start by examining the case where there is no dashpot nor any other form of resistance, so the only force applied to the mass is exerted by the spring. The differential equation describing the mass's displacement from the equilibrium position is thus

$$mx''(t) + kx(t) = 0 \tag{3.1.1}$$

where k is the spring constant. It shall be beneficial to rewrite the equation as

$$x''(t) + \omega_0^2 x(t) = 0$$

where the quantity

$$\omega_0 = \sqrt{\frac{k}{m}}$$

will play, as we shall shortly see, a crucial role in the understanding of the motion.

The characteristic equation of the second-order linear homogeneous differential equation (3.1.1) is

$$r^2 + \omega_0^2 = 0 ,$$

admitting a pair of complex-conjugate solutions

$$r_1 = i\omega_0 , \quad r_2 = -i\omega_0 .$$

The general solution is thus given by

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) , \quad A, B \in \mathbb{R}.$$

It is possible to recast the solution using only one trigonometric function, instead of two. To this effect, we introduce two new real parameters C, α, subject to the relations

$$C = \sqrt{A^2 + B^2} , \quad \cos\alpha = \frac{A}{C} , \quad \sin\alpha = \frac{B}{C} , \quad \alpha \in [0, 2\pi) .$$

Using the new parameters, we may rewrite the general solution, at least when A and B are not both 0[1], as

$$x(t) = C\left(\frac{A}{C}\cos(\omega_0 t) + \frac{B}{C}\sin(\omega_0 t)\right) = C(\cos\alpha\cos(\omega_0 t) + \sin\alpha\sin(\omega_0 t)) .$$

Recalling now the addition formula for the cosine,

$$\cos(\beta + \gamma) = \cos\beta\cos\gamma - \sin\beta\sin\gamma ,$$

we can write from above

$$x(t) = C\cos(\omega_0 t - \alpha) .$$

Remark 3.1.1. Observe that, although to derive this alternative expression we had to assume $(A, B) \neq (0, 0)$, we actually recover this trivial case by letting $C = 0$ in the last displayed expression for $x(t)$. Thus, the latter describes fully the general solution to (3.1.1).

[1] This corresponds to the uninteresting case where the initial position of the mass is the equilibrium position, and the initial velocity vanishes, so that the mass stays put forever.

Suppose the mass has initial position and velocity

$$x(0) = x_0 \ , \quad v(0) = v_0 \ ;$$

then we get

$$x_0 = C\cos(\omega_0 \cdot 0 - \alpha) = C\cos(-\alpha) = C\cos\alpha \ ,$$

and computing

$$x'(t) = -\omega_0 C\sin(\omega_0 t - \alpha) \ ,$$

we get also the condition

$$v_0 = -\omega_0 C\sin(\omega_0 \cdot 0 - \alpha) = -\omega_0 C\sin(-\alpha) = \omega_0 C\sin\alpha \ .$$

The conditions

$$\begin{cases} C\cos\alpha = x_0 \\ C\sin\alpha = \frac{v_0}{\omega_0} \end{cases}$$

are met for

$$C = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}$$

3.1.2. Free damped motion.

$$mx''(t) + cx'(t) + kx(t) = 0 \ , \tag{3.1.2}$$

which is convenient to rewrite as

$$x''(t) + 2px'(t) + \omega_0^2 x(t) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{m}}$$

being the associated undamped circular frequency and

$$p = \frac{c}{2m} > 0 \ .$$

The characteristic equation of the second-order linear homogeneous differential equation (3.1.2) is

$$r^2 + 2pr + \omega_0^2 = 0 \ ,$$

which has roots

$$r_{1,2} = -p \pm \sqrt{p^2 - \omega_0^2} \ . \tag{3.1.3}$$

The analysis of solutions to (3.1.2) depends thus on whether we are in the regime

$$p^2 > \omega_0^2 \ , \quad p^2 = \omega_0^2 \ , \quad \text{or} \quad p^2 < \omega_0^2 \ .$$

The quantity under the square-root sign in (3.1.3) is, explicitly,

$$p^2 - \omega_0^2 = \frac{c^2}{4m^2} - \frac{k}{m} = \frac{c^2 - 4km}{4m^2} \ ; \tag{3.1.4}$$

the **critical damping** is defined as

$$c_{\mathrm{cr}} = \sqrt{4km} \ ,$$

whence the three regimes mentioned above correspond to the cases

$$c > c_{\mathrm{cr}} \ , \quad c = c_{\mathrm{cr}} \ , \quad c < c_{\mathrm{cr}} \ .$$

Overdamped case: $c > c_{\mathbf{cr}}$. This is the regime where the resistance (represented by c) is strong compared to the force the spring is capable of exerting (represented by k). The characteristic equation of (3.1.2) has two distinct real roots

$$r_1 = -p + \sqrt{p^2 - \omega_0^2}\ , \quad r_2 = -p - \sqrt{p^2 - \omega_0^2}\ ,$$

both of which are strictly negative (since $\sqrt{p^2 - \omega_0^2} < p$), and therefore admits as general solution

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}\ , \quad c_1, c_2 \in \mathbb{R}.$$

We deduce that, regardless of the initial position and velocity of the object, or equivalently, regardless of the constants c_1 and c_2,

$$\lim_{t\to\infty} x(t) = 0\ .$$

Thus, the position of the object will converge exponentially fast to its equilibrium $x = 0$: the would-be oscillations caused by the spring are completely damped out.

Critically damped case: $c = c_{\mathbf{cr}}$. We might think of this situation as the case where the resistance and the force imparted by the spring are fully balanced out. In this case, the characteristic equation of (3.1.2) has one multiple real root

$$r = -p\ ;$$

this yields the general solution

$$x(t) = c_1 e^{-pt} + c_2 t e^{-pt} = e^{-pt}(c_1 + c_2 t)\ , \quad c_1, c_2 \in \mathbb{R}.$$

Observe that, here, the equation

$$x(t) = 0$$

admits, if $c_2 \neq 0$, exactly one solution in t, namely

$$t = -\frac{c_1}{c_2}\ ;$$

hence, if $c_1/c_2 < 0$, the object will pass exactly once through its equilibrium position, at time $-\frac{c_1}{c_2} > 0$, and subsequently converge exponentially fast to it.

Underdamped case: $c < c_{\mathbf{cr}}$. This last scenario corresponds to the case where the resistance is not enough to dampen the force exerted by the spring. The characteristic equation of (3.1.2) has two complex-conjugate roots

$$r_1 = -p + i\sqrt{\omega_0^2 - p^2}\ ; \quad r_2 = -p - i\sqrt{\omega_0^2 - p^2}\ ,$$

and thus admits as general solution

$$x(t) = e^{-pt}(A\cos(\omega_1 t) + B\sin(\omega_1 t))$$

where we set

$$\omega_1 = \sqrt{\omega_0^2 - p^2} = \frac{1}{2m}\sqrt{4km - c^2}\ ,$$

the last equality following from (3.1.4).

Introducing now two new parameters C, α via

$$C = \sqrt{A^2 + B^2}\ , \quad \cos\alpha = \frac{A}{C}\ , \quad \sin\alpha = \frac{B}{C}\ ,$$

and using the addition formula for the cosine, as we did in §3.1.1, we can rewrite the general solution as

$$x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)\ .$$

We call

$$\omega_1$$

the **pseudofrequency**,

$$T_1 = \frac{2\pi}{\omega_1}$$

the **pseudoperiod** and

$$Ce^{-pt}$$

the **time-varying amplitude** of the oscillation.

3.2. Nonhomogeneous equations and undetermined coefficients

The general form of a non-homogeneous linear ODE of order n, in the unknown $y = y(x)$, is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \ , \tag{3.2.1}$$

where f is a continuous function of a real variable x, referred to as the *non-homogeneous term* of the equation, and $a_0, a_1, \ldots, a_n$ are given real numbers with $a_n \neq 0$.

THEOREM 3.2.1. *Consider a differential equation as in (3.2.1), and let $y_p(x)$ be a particular solution. Let also*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{3.2.2}$$

be the associated homogeneous differential equation.

(1) If $y_h(x)$ is a solution of (3.2.2), then

$$y(x) = y_h(x) + y_p(x)$$

is a solution of (3.2.1).

(2) Conversely, if $y(x)$ is a solution of (3.2.1), then there exists a solution $y_h(x)$ of (3.2.2) such that

$$y(x) = y_h(x) + y_p(x) \ .$$

Method of undetermined coefficients. Suppose we want to find a particular solution to

$$y''(x) + ay'(x) + by(x) = A\cos kx + B\sin kx \ , \tag{3.2.3}$$

where a, b, A, B, k are given real numbers with $k \neq 0$. We may assume that $b \neq 0$, otherwise the equation boils down to a first-order equation in the new unknown function $u(x) = y'(x)$.

We look for a particular solution of the form

$$y_p(x) = \alpha\cos kx + \beta\sin kx \ , \tag{3.2.4}$$

for some real coefficients α, β to be determined. We compute

$$y'(x) = k(-\alpha\sin kx + \beta\cos kx)$$

and

$$y''(x) = -k^2(\alpha\cos kx + \beta\sin kx) = -k^2 y(x) \ .$$

Therefore, for the homogeneous term we have

$$y''(x)+ay'(x)+by(x) = (b-k^2)y(x)+ay'(x) = \big((b-k^2)\alpha+ak\beta\big)\cos kx+\big((b-k^2)\beta-ak\alpha\big)\sin kx \ .$$

In order for (3.2.4) to be a particular solution of (3.2.3), we must therefore have

$$\big((b-k^2)\alpha + ak\beta\big)\cos kx + \big((b-k^2)\beta - ak\alpha\big)\sin kx = A\cos kx + B\sin kx \ . \tag{3.2.5}$$

Since $\cos kx$ and $\sin kx$ are linearly independent functions, we must have

$$\begin{cases} (b-k^2)\alpha + ak\beta = A \\ (b-k^2)\beta - ak\alpha = B \end{cases} . \tag{3.2.6}$$

It is a classical result from elementary linear algebra that the previous linear system admits a unique solution in the unknowns α and β if and only if the determinant of the matrix

$$\begin{pmatrix} b-k^2 & ak \\ -ak & b-k^2 \end{pmatrix}$$

is non-zero. Such determinant equals

$$(b - k^2)^2 + (ak)^2 ,$$

which is a strictly positive real number unless

$$a = 0 \quad \text{and} \quad b = k^2$$

(recall we operate under the assumption $k \neq 0$).

Therefore, if either $a \neq 0$ or $b \neq k^2$, then there is a unique pair (α, β) of real numbers such that the function

$$y_p(x) = \alpha \cos kx + \beta \sin kx$$

is a particular solution to (3.2.3).

Suppose now we are in the pathological case where both $a = 0$ and $b = k^2$, so that (3.2.3) takes the form

$$y'' + k^2 y = A \cos kx + B \sin kx \ . \tag{3.2.7}$$

It is clear in this case why no function of the form

$$y_p(x) = \alpha \cos kx + \beta \sin kx \tag{3.2.8}$$

can be a solution (unless $A = B = 0$, and then we are in the homogeneous situation which we already understand fully); indeed, the associated homogeneous equation is

$$y'' + k^2 y = 0 \ , \tag{3.2.9}$$

and since the roots of the corresponding characteristic equation are $\pm ik$, the general solution of (3.2.9) is precisely given by (3.2.8), which can thus not be a solution to the non-homogeneous equation. We thus attempt to find a particular solution to (3.2.7) by slightly tweaking (3.2.8), and looking for solutions of the form

$$y_p(x) = x(\alpha \cos kx + \beta \sin kx) \ .$$

We compute

$$y_p'(x) = \alpha \cos kx + \beta \sin kx + kx(-\alpha \sin kx + \beta \cos kx)$$

and

$$\begin{aligned} y_p''(x) &= 2k(-\alpha \sin kx + \beta \cos kx) - k^2 x(\alpha \cos kx + \beta \sin kx) \\ &= 2k(-\alpha \sin kx + \beta \cos kx) - k^2 y_p(x) \ . \end{aligned}$$

Therefore,

$$y_p''(x) + k^2 y_p(x) = 2k(-\alpha \sin kx + \beta \cos kx) = 2k\beta \cos kx - 2k\alpha \sin kx \ ,$$

and we want the latter expression to equal

$$A \cos kx + B \sin kx$$

in order for y_p to be a solution to the non-homogeneous equation (3.2.7). Since $k \neq 0$, $\cos kx$ and $\sin kx$ are linearly independent functions, and thus the only way the desired equality can hold is to have

$$\begin{cases} 2k\beta = A \\ -2k\alpha = B \end{cases} .$$

We deduce thus that the function

$$y_p(x) = x\left(-\frac{B}{2k} \cos kx + \frac{A}{2k} \sin kx\right)$$

is a particular solution of (3.2.7).

EXAMPLE 3.2.2. Let's solve the initial value problem

$$y'' + 4y' - 12y = 5\cos 3x\ , \quad y(0) = -\frac{7}{39}\ ,\ y'(0) = \frac{17}{13}$$

The general solution to the nonhomogeneous equation

$$y'' + 4y' - 12y = 5\cos 3x$$

is of the form

$$y(x) = y_p(x) + y_h(x)$$

where $y_p(x)$ is a particular solution of the equation and $y_h(x)$ is the general solution of the associated homogeneous equation

$$y'' + 4y' - 12y = 0\ .$$

The characteristic equation of the latter is

$$r^2 + 4r - 12 = (r+6)(r-2) = 0\ ,$$

whence the general solution takes the form

$$y_h(x) = Ae^{-6x} + Be^{2x}\ , \quad A, B \in \mathbb{R}.$$

We now want to find a particular solution y_p of the non-homogeneous equation. Since the coefficient of the term y' appearing in the equation is $4 \neq 0$, we may look for a particular solution of the form

$$y_p(x) = \alpha\cos 3x + \beta\sin 3x$$

for real parameters α and β to be determined. We compute

$$y_p'(x) = -3\alpha\sin 3x + 3\beta\cos 3x\ ,$$

$$y_p''(x) = -9(\alpha\cos 3x + \beta\sin 3x)\ ,$$

and

$$y_p''(x) + 4y_p'(x) - 12y_p(x) = (-21\alpha + 12\beta)\cos 3x - (12\alpha + 21\beta)\sin 3x\ .$$

In order to find α and β, we must therefore solve the linear system

$$\begin{cases} -21\alpha + 12\beta = 5 \\ 12\alpha + 21\beta = 0 \end{cases} .$$

From the second equation we derive

$$\beta = -\frac{4}{7}\alpha\ ,$$

and plugging this into the first equation we deduce

$$-21\alpha + 12\left(-\frac{4}{7}\alpha\right) = 5\ ,$$

that is,

$$\alpha = -\frac{7}{39}\ , \quad \beta = \frac{4}{39}\ .$$

All in all, we have found that the general solution to the given non-homogeneous equation is

$$y(x) = \frac{1}{39}(-7\cos 3x + 4\sin 3x) + Ae^{-6x} + Be^{2x}\ , \quad A, B \in \mathbb{R}.$$

To determine A and B, all is left to do is to impose the given initial conditions. We have

$$-\frac{7}{39} = y(0) = -\frac{7}{39} + A + B\ , \quad \text{that is,} \quad A + B = 0\ ;$$

for the derivative, we have

$$y'(x) = \frac{1}{13}(7\sin 3x + 4\cos 3x) - 6Ae^{-6x} + 2Be^{2x}\ ,$$

thus we want

$$\frac{17}{13} = y'(0) = \frac{4}{13} - 6A + 2B \ , \quad \text{that is,} \quad -6A + 2B = 1 \ .$$

We thus need to solve the linear system

$$\begin{cases} A + B = 0 \\ -6A + 2B = 1 \end{cases} ,$$

which is easily seen to admit the unique solution

$$A = -\frac{1}{8} \ , \quad B = \frac{1}{8} \ .$$

Hence, the unique solution to the given IVP is

$$y(x) = \frac{1}{39}(-7\cos 3x + 4\sin 3x) + \frac{1}{8}(e^{2x} - e^{-6x}) \ .$$

3.3. Endpoint problems and eigenvalues

Much of the discussion in the present chapter relies fundamentally on the existence and uniqueness of solutions for initial value problems associated with linear second order differential equations (Theorem **??**), namely the fact that an IVP of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \ , \quad y(0) = a \ , \quad y'(0) = b$$

admits, *generically* (that is, under extremely mild regularity assumptions on the functions p and q), a unique solution. The conclusive section of this chapter introduces a different sort of problems, for which existence and uniqueness of solutions may instead fail dramatically. These are called **endpoint problems** or **boundary value problems**, and in the context of linear second-order differential equations take the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \ , \quad y(a) = 0 \ , \quad y(b) = 0 \ ,$$

where $p(x)$ and $q(x)$ are continuous functions of one real variable, $a < b$ are real numbers, and we look for solutions $y(x)$ which are continuous functions $y(x)$ defined on the closed interval $[a, b]$, twice continuously differentiable on the open interval (a, b), therein satisfying the given differential equation, and finally satisfying the prescribed boundary conditions $y(a) = y(b) = 0$.

A boundary value problem of the form

$$y''(x) + p(x)y'(x) + \lambda q(x)y(x) = 0 \ , \quad y(a) = 0 \ , \quad y(b) = 0 \ , \tag{3.3.1}$$

where λ is an unspecified real parameter, is called an **eigenvalue problem**. The question we would like to answer regarding this kind of problems is the following: for which values of λ does the boundary value problem admit a non-trivial solution, namely a solution $y(x)$ which is not constantly equal to zero[2]? If such a nonzero solution exists, we say that λ is an **eigenvalue** of the problem.

If λ_* is an eigenvalue of the problem (3.3.1) and y_* is a corresponding nonzero solution to the problem, namely the conditions

$$y_*'' + p(x)y_*' + \lambda_* q(x)y_* = 0 \ , \quad y_*(a) = 0 \ , \quad y_*(b) = 0$$

are fulfilled, we call y_* an **eigenfunction** associated to the eigenvalue λ_*.

Observe that the problem (3.3.1) is *homogeneous*, namely whenever y_* is an eigenfunction associated to a certain eigenvalue λ_*, then any constant multiple cy_*, $0 \neq c \in \mathbb{R}$, is also an eigenfunction associated to the same eigenvalue.

[2]Observe that the constant function $y = 0$ satisfies trivially the conditions dictated by the eigenvalue problem

Remark 3.3.1. Under mild regularity assumptions on the functions $p(x)$ and $q(x)$, it can be proven that, conversely, any two eigenfunctions associated to the same eigenvalue λ_* for the problem (3.3.1) must be linearly dependent. However, we won't need such an abstract result in the sequel, it will emerge on a case-by-case basis through computations.

We shall now see how to find eigenvalues and eigenfunctions for eingevalue problems in concrete examples.

Example 3.3.2. Let's determine eigenvalues and associated eigenfunctions for the boundary value problem

$$y'' + \lambda y = 0 \ , \quad y(0) = 0 \ , \quad y(L) = 0$$

where $L > 0$ is a fixed real number.

We start by determining the general solution to the given ODE. The characteristic equation is

$$r^2 + \lambda = 0 \ ,$$

whence we need to examine three distinct cases separately, according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$.

Suppose first $\lambda > 0$. Then the characteristic equations has two complex conjugate roots

$$r_{1,2} = \pm i\sqrt{\lambda} \ ,$$

whence the general solution to the ODE is given by

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \ , \quad A, B \in \mathbb{R}.$$

We now need to impose the boundary conditions: we have

$$0 = y(0) = A$$

and

$$0 = y(L) = A\cos(\sqrt{\lambda}L) + B\sin(\sqrt{\lambda}L) \ .$$

If $\sin(\sqrt{\lambda}L) \neq 0$, that is, if

$$\lambda \neq k^2\,\frac{\pi^2}{L^2} \quad \text{for all integer } k \ ,$$

then the only possibility is to have

$$A = B = 0 \ ,$$

so that the unique solution to the boundary value problem for such a λ is the zero function. In this case, therefore, λ is not an eigenvalue of the problem. If, instead,

$$\lambda = k^2\,\frac{\pi^2}{L^2}$$

for some integer $k \neq 0$ (recall that we are in the case $\lambda > 0$), then $\sin(\sqrt{\lambda}L) = 0$, and from the above we see that any function

$$y(x) = B\sin(\sqrt{\lambda}x) \ , \quad B \neq 0$$

is a nonzero solution to the eigenvalue problem for such a value of λ. We conclude that $\lambda_* = k^2\,\frac{\pi^2}{L^2}$ is an eigenvalue for our problem, for any integer $k \neq 0$, and $y_*(x) = B\sin(\sqrt{\lambda_*}x)$ is an associated eigenfunction for any $B \neq 0$.

We proceed with the case $\lambda = 0$. The general solution to the differential equation is then

$$y(x) = Ax + B \ , \quad A, B \in \mathbb{R},$$

and it is clear that the conditions $y(0) = 0 = y(L)$ can only be both met if $A = B = 0$. Thus the unique solution to the boundary value problem corresponding to $\lambda = 0$ is the zero solution, which tells us that 0 is not an eigenvalue for the problem.

Finally, we examine the case $\lambda < 0$. In this case the characteristic equation associated to the differential equation has two distinct real roots,

$$r_{1,2} = \pm\sqrt{-\lambda} \ ,$$

whence the general solution of the ODE takes the form

$$y(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}\,, \quad A, B \in \mathbb{R}.$$

In order for the boundary conditions $y(0) = 0 = y(L)$ to be fulfilled, the parameters A and B must solve the linear system

$$\begin{cases} A + B = 0 \\ e^{\sqrt{-\lambda}L}A + e^{-\sqrt{-\lambda}L}B = 0 \end{cases} \quad . \tag{3.3.2}$$

The determinant of the 2×2 matrix

$$\begin{pmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}L} & e^{-\sqrt{-\lambda}L} \end{pmatrix}$$

is

$$e^{-\sqrt{-\lambda}L} - e^{\sqrt{-\lambda}L}\,,$$

which vanishes if and only if

$$e^{\sqrt{-\lambda}L} = e^{-\sqrt{-\lambda}L}\,,$$

that is, if and only if

$$\sqrt{-\lambda}L = 0\,.$$

Since $L > 0$ by assumption, the only possibility would be $\lambda = 0$, which is however excluded since we are dealing with the case $\lambda < 0$.

It follows that the determinant is always nonvanishing, which implies by elementary linear algebra that the linear system (4.1.2) admits only the trivial solution $A = B = 0$. Thus, the only function solving the boundary value problem is the function $y = 0$, which again tells us that no $\lambda < 0$ can be an eigenvalue for our given problem.

Observe that the previous example illustrates already the difference between boundary value problems and initial value problems in terms of uniqueness of solutions. When

$$\lambda = k^2\frac{\pi^2}{L^2}$$

for some integer $k \neq 0$, then the boundary value problem

$$y'' + \lambda y = 0\,, \quad y(0) = 0\,, \quad y(L) = 0 \quad (L > 0)$$

admits infinitely many solutions, namely all functions of the form

$$y(x) = B\sin\left(\sqrt{\lambda}x\right)\,, \quad B \in \mathbb{R}.$$

Boundary value problems may also involve conditions on the first derivative, potentially mixed with conditions on the unknown function itself. For instance, they may take the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0\,, \quad y'(a) = 0\,, \quad y'(b) = 0$$

or

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0\,, \quad y(a) = 0\,, \quad y'(b) = 0\,.$$

Let's investigate eigenvalue problems of this sort.

EXAMPLE 3.3.3. We treat the eigenvalue problem

$$y'' + \lambda y = 0\,, \quad y(0) = 0\,, \quad y'(L) = 0$$

for some fixed $L > 0$. The differential equation is the same as in Example 3.3.2, and we similarly distinguish the three cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

When $\lambda > 0$, the general solution to the differential equation is

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)\,, \quad A, B \in \mathbb{R}.$$

The derivative is

$$y'(x) = \sqrt{\lambda}(-A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x))\,,$$

whence imposing the boundary conditions $y(0) = 0 = y'(L)$ amounts to having

$$A = 0\ , \quad \sqrt{\lambda} B \cos(\sqrt{\lambda} L) = 0\ .$$

As $\lambda > 0$, the last displayed condition yields

$$B \cos(\sqrt{\lambda} L) = 0\ . \tag{3.3.3}$$

Now $\cos(\sqrt{\lambda} L) = 0$ holds if and only if

$$\lambda = \frac{\pi^2}{4L^2}(1+2k)^2$$

for some integer k. We infer that the value

$$\lambda_* = \frac{\pi^2}{4L^2}(1+2k)^2$$

is an eigenvalue of our problem for every integer k, with associated family of eigenfunctions

$$y_*(x) = B \sin(\sqrt{\lambda_*} x)\ , \quad B \neq 0.$$

On the other hand, if $\lambda > 0$ is different from $\frac{\pi^2}{4L^2}(1+2k)^2$ for all integers k, then the only way (3.3.3) can hold is for $B = 0$, which leads to $y = 0$ as the only solution to the corresponding boundary value problem. Such values of λ aren't thus eigenvalues for the problem.

When $\lambda = 0$, the general solution to the differential equation is

$$y(x) = Ax + B\ , \quad A, B \in \mathbb{R},$$

with derivative

$$y'(x) = A\ .$$

The boundary conditions thus deliver

$$0 = y(0) = B\ , \quad 0 = y'(L) = A\ ,$$

from which we deduce that 0 is not an eigenvalue for our problem.

Finally, when $\lambda < 0$, the general solution to the differential equation is given by

$$y(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}\ , \quad A, B \in \mathbb{R},$$

with derivative

$$y'(x) = \sqrt{-\lambda} A e^{\sqrt{-\lambda}x} - \sqrt{-\lambda} B e^{-\sqrt{-\lambda}x}\ .$$

The boundary conditions $y(0) = 0 = y'(L)$ result thus in the linear system

$$\begin{cases} A + B = 0 \\ \sqrt{-\lambda} e^{\sqrt{-\lambda}L} A - \sqrt{-\lambda} e^{-\sqrt{-\lambda}L} B = 0 \end{cases} .$$

The determinant of the matrix

$$\begin{pmatrix} 1 & 1 \\ \sqrt{-\lambda} e^{\sqrt{-\lambda}L} & -\sqrt{-\lambda} e^{-\sqrt{-\lambda}L} \end{pmatrix}$$

vanishes if and only if

$$-\sqrt{-\lambda} e^{\sqrt{-\lambda}} = \sqrt{-\lambda} e^{-\sqrt{-\lambda}}\ ,$$

which is impossible since $\lambda \neq 0$. Thus no $\lambda < 0$ can be an eigenvalue of our problem.

CHAPTER 4

Linear systems of differential equations

The purpose of this chapter is to introduce the study of *systems of differential equations.* Just like, in elementary algebra, we pass from the study of a single algebraic equation to the study of systems of such equations, such as the linear system

$$\begin{cases} 3X_1 + 4X_2 = 1 \\ -2X_1 + 5X_2 = 0 \end{cases}$$

in the two variables X_1, X_2, so too in the context of differential equations we might be interested, for the purpose of understanding real-world phenomena, in studying systems of differential equations, namely finite collections of differential equations in several different unknown functions, such as the *linear system*

$$\begin{cases} x_1'(t) = 2x_2(t) \\ x_2'(t) = -3x_1(t) + e^t \end{cases} .$$

As we shall see in the first section if this chapter, a linear system of n first-order differential equations in n unknown functions $x_1(t), \dots, x_n(t)$ takes the form

$$\begin{cases} x_1'(t) = p_{11}(t)x_1(t) + p_{12}(t)x_2(t) + \cdots + p_{1n}(t)x_n(t) + f_1(t) \\ x_2'(t) = p_{21}(t)x_1(t) + p_{22}(t)x_2(t) + \cdots + p_{2n}(t)x_n(t) + f_2(t) \\ \cdots \\ x_n'(t) = p_{n1}(t)x_1(t) + p_{n2}(t)x_2(t) + \cdots + p_{nn}(t)x_n(t) + f_n(t) \end{cases}$$

where $p_{11}(t), \dots, p_{1n}(t), \dots, p_{n1}(t), \dots, p_{nn}(t)$ and $f_1(t), \dots, f_n(t)$ are given real-valued functions of a single real variable t. It is already apparent how cumbersome the previous notation for a general linear system of n differential equations can be. However, linear algebra comes to the rescue and allows to simplify notation considerably, allowing for instance to express the last displayed linear system in a much more concise form as

$$x'(t) = P(t)x(t) + f(t) \; ,$$

as if it was a single linear first-order differential equation, where now $x(t)$ and $f(t)$ are *column vector-valued functions* and $P(t)$ is a *matrix-valued function.* We will thus begin the chapter with a self-contained review of matrix terminology and notation.

4.1. Matrices and linear systems

Given two integers $m, n \geq 1$, an $m \times n$ **matrix** is a rectangular array consisting of mn real numbers, also known es **elements** or **entries** of the matrix, arranged in m (horizontal) **rows** and n (vertical) **columns**:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{pmatrix} .$$

We will routinely denote a matrix A as above, in abbreviated form, as $A = [a_{ij}]$, where a_{ij} is the element in the i-th row and j-th column.

If $m = n$, we say that an $n \times n$ matrix A is a **square matrix** of order n.

By definition, two matrices $A = [a_{ij}]$, $B = [b_{ij}]$ are **equal** if they have the same number m of rows and the same number n of columns, and if $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

4.1.1. Addition and scalar multiplication. We introduce two operations on the set of $m \times n$ matrices. First, given two $m \times n$ matrices $A = [a_{ij}]$, $B = [b_{ij}]$, we define the **sum** of A and B as the $m \times n$ matrix

$$A + B = [a_{ij} + b_{ij}] ;$$

in other words, the element in row i and column j of the sum $A + B$ is the sum $a_{ij} + b_{ij}$ of the corresponding elements in A and B, respectively.

Secondly, given a real number c, we define the **scalar multiplication** of an $m \times n$ matrix $A = [a_{ij}]$ by the scalar c as the $m \times n$ matrix

$$cA = [ca_{ij}] ,$$

obtained by multiplying each element of A by c.

The two operations just defined satisfy the following properties:

- (associativity of the sum) if A, B, C are $m \times n$ matrices, then
$$(A + B) + C = A + (B + C) ;$$
- (commutativity of the sum) if A, B are $m \times n$ matrices, then
$$A + B = B + A ;$$
- (existence of additive identity) the **zero** $m \times n$ **matrix** $\mathbf{0}$, namely the $m \times n$ matrix all of whose elements are zero, is the identity for the sum: for any $m \times n$ matrix,
$$\mathbf{0} + A = A + \mathbf{0} = A ;$$
- (existence of additive inverse) for any $m \times n$ matrix $A = [a_{ij}]$, the matrix $-A = [-a_{ij}]$ is an additive inverse for A, namely
$$A + (-A) = -A + A = \mathbf{0} ;$$
- (distributivity of scalar multiplication with respect to the sum) if A, B are $m \times n$ matrices and c, d are real numbers, then
$$(c + d)A = cA + dA \quad \text{and} \quad c(A + B) = cA + cB ;$$
- (existence of identity for scalar multiplication) for every $m \times n$ matrix A,
$$1A = A ,$$
where $1A$ denotes scalar multiplication of A by the real number 1.

The **transpose** of an $m \times n$ matrix $A = [a_{ij}]$, which we will indicate with A^T, is the $n \times m$ matrix $A^T = [a_{ji}]$: more precisely, the element in the j-th row and i-th colum of A^T is the element in the i-th row and j-th column of A.

An $m \times 1$ matrix is called a **column vector**, whereas an $1 \times n$ matrix is called a **row vector**. The transpose of a row vector is a column vector, and conversely.

It is often convenient to describe an $m \times n$ matrix $A = [a_{ij}]$ as an array of m row vectors

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{pmatrix}$$

or as an array of n column vectors

$$A = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

where

$$a_i = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix}$$

for every $1 \leq i \leq m$ and

$$b_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{pmatrix}$$

for every $1 \leq j \leq n$.

4.1.2. Matrix multiplication. Another very important operation we can perform on matrices is **matrix multiplication**. We first introduce the multiplication of a row vector by a column vector. If $a = (a_1\ a_2\ \cdots\ a_n)$ is a $1 \times n$ row vector and $b = (b_1\ b_2\ \cdots\ b_n)^T$ is an $n \times 1$ column vector, then the matrix multiplication of a by b, denoted ab, is the real number

$$ab = \sum_{k=1}^{n} a_k b_k = a_1 b_1 + \cdots + a_n b_n \ ,$$

that is, the familiar *dot product* of the two vectors a and b.

If now A and B are two matrices, *where the number of columns of A matches the number of rows of B*[1], we define the product AB as follows. If $A = [a_{rs}]$ is an $m \times \ell$ matrix and $B = [b_{tu}]$ is an $\ell \times n$ matrix, the product $AB = [c_{ij}]$ is the $m \times n$ matrix where

$$c_{ij} = \sum_{k=1}^{\ell} a_{ik} b_{kj} = a_{i1} b_{1j} + \cdots + a_{i\ell} b_{\ell j}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$. In other words, the element c_{ij} of the product AB is the scalar product, or dot product, of the i-th row vector of A with the j-th column vector of B:

$$c_{ij} = a_i \cdot b_j$$

where we write

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} \ .$$

It is straightforward, though tedious, to verify that matrix multiplication is associative:

$$A(BC) = (AB)C$$

whenever all products are defined. Furthermore, it distributes with respect to the sum, meaning that

$$A(B + C) = AB + AC \quad \text{and} \quad (B + C)A = BA + CA$$

whenever the products are defined. Also, there is an identity for matrix multiplication on square matrices: if $\mathbf{I}$ denotes the **identity matrix** of order n, namely the matrix with 1 in the diagonal entries and 0 in the off-diagonal entries, then

$$\mathbf{I}A = A\mathbf{I} = A$$

for every square matrix A of order n.

[1]Else, matrix multiplication is not defined.

Observe, however, that matrix multiplication is *not* commutative. To begin with, it is not necessarily true that the product BA is well defined whenever AB is: this only happens for square matrices. However, even confining ourselves to square matrices of a fixed order, $AB \neq BA$ in general.

EXAMPLE 4.1.1. Consider the 2×2 matrices

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ;$$

then

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{but} \quad BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

Notice also that, in contrast to what happens for standard multiplication of real numbers, it may happen for two (say, square) matrices A, B that

$$AB = \mathbf{0} \quad \text{with} \quad A \neq \mathbf{0} \quad \text{and} \quad B \neq \mathbf{0}.$$

EXAMPLE 4.1.2. Consider the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ,$$

which are both different from the zero 2×2 matrix $\mathbf{0}$. However,

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0} .$$

4.1.3. Inverse matrices and determinants. We say that a square matrix A, of order n, admits a **multiplicative inverse** if there is a square matrix B, of order n, such that

$$AB = BA = \mathbf{I} .$$

It is straightforward to prove that, if A admits a multiplicative inverse, then such inverse is unique[2]; we denote it by A^{-1}.

An important fact from elementary linear algebra is that a square matrix A is **invertible**, namely admits a multiplicative inverse, if and only if its **determinant** $\det A$, also denoted $|A|$ in the sequel, doesn't vanish. Recall that the determinant of a square matrix can be defined, inductively on the order n of the matrix, as follows. For 2×2 matrices, the determinant is the difference between the product of the diagonal elements and the product of the anti-diagonal elements: if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

then

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc .$$

---

[2]Here is the easy argument: if B and B' multiplicative inverses for A, then we may write the following chain of equalities:

$$B = \mathbf{I}B = (B'A)B = B'(AB) = B'\mathbf{I} = B' ,$$

where we have used, in successive order:

- the fact that $\mathbf{I}$ is a multiplicative identity;
- the fact that B' is a multiplicative inverse for A;
- associativity of matrix multiplication;
- the fact that B is a multiplicative inverse for A;
- the fact that $\mathbf{I}$ is a multiplicative identity.

Assume now we know how to compute determinants of square matrices of order $n-1$; then the *expansion of the determinant of a square matrix* $A = [a_{ij}]$ *of order* n *along the* i*-th row*, where i is some fixed integer between 1 and n, is

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \ ,$$

where $|A_{ij}|$ is the determinant of the square matrix A_{ij} of order $n-1$ obtained by deleting the i-th row and the j-th column from the matrix A. Similarly, the expansion of the determinant of A along the j-th column, where i is some fixed integer between 1 and n, is

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \ .$$

It is known from linear algebra that all such sums yield the same value, for all $1 \le i \le n$ and all $1 \le j \le n$, which is by definition the determinant of the square matrix A.

Example 4.1.3. Consider the 3×3 matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 4 \\ 1 & 0 & 1 \end{pmatrix} .$$

We compute its determinant using expansion along the first column. We have

$$|A| = 2 \begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix} = 2(-1) + 1 + 4 = 3 \ .$$

Alternatively, we may compute it, say, using expansion along the first row:

$$|A| = 2 \begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 4 \\ 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} = 2(-1) - (-1-4) + 0 = 3 \ .$$

4.1.4. Matrix-valued functions. By a **matrix-valued function** we mean a function defined on some open interval $I \subset \mathbb{R}$ taking values in the set of $m \times n$ matrices, where $m, n \ge 1$ are fixed integers. Indicating with t the independent real variable, ranging over the interval I, we typically write a matrix-valued function as

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \cdots & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) & \cdots & \cdots & a_{2n}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) & \cdots & \cdots & a_{3n}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}(t) & a_{m2}(t) & a_{m3}(t) & \cdots & \cdots & a_{mn}(t) \end{pmatrix}$$

or, in shorthand form, $A(t) = [a_{ij}(t)]$, where $a_{ij}(t)$ is a real-valued function defined on the open interval I, for all $1 \le i \le m$ and all $1 \le j \le n$.

We say that a matrix-valued function $A(t) = [a_{ij}(t)]$ is **continuous** (respectively, **differentiable**) if all functions $a_{ij}(t)$ are continuous (respectively, differentiable). If $A(t)$ is differentiable, its **derivative** is the matrix-valued function

$$A'(t) = [a'_{ij}(t)] \ ,$$

namely the matrix of derivatives of the elements of $A(t)$.

Example 4.1.4. If

$$A(t) = \begin{pmatrix} 1 & 0 & \cos t \\ \sin t & e^{-t} & 1 + t^2 \end{pmatrix} ,$$

then $A(t)$ is differentiable with derivative

$$A'(t) = \begin{pmatrix} 0 & 0 & -\sin t \\ \cos t & -e^{-t} & 2t \end{pmatrix} .$$

Here is how differentiation of matrix-valued functions behaves with respect to the standard matrix operations:

- if $A(t)$ and $B(t)$ are differentiable matrix-valued functions, then $A(t)+B(t)$ is a differentiable matrix-valued function and
$$\frac{\mathrm{d}}{\mathrm{d}t}\big(A(t)+B(t)\big)=\frac{\mathrm{d}}{\mathrm{d}t}A(t)+\frac{\mathrm{d}}{\mathrm{d}t}B(t)\ ;$$
- if $A(t)$ is a differentiable matrix-valued function and $c(t)$ is a differentiable real-valued function, then the scalar multiplication $c(t)A(t)$ is a differentiable matrix-valued function and
$$\frac{\mathrm{d}}{\mathrm{d}t}\big(c(t)A(t)\big)=\left(\frac{\mathrm{d}}{\mathrm{d}t}c(t)\right)A(t)+c(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)\ ;$$
- if $A(t)$ is a differentiable matrix-valued function with values in the set of $m\times\ell$ matrices, and $B(t)$ is a differentiable matrix-valued function with values in the set of $\ell\times n$ matrices, then the matrix product $A(t)B(t)$ is (well defined and) a differentiable matrix-valued function with values in the set of $m\times n$ matrices, and
$$\frac{\mathrm{d}}{\mathrm{d}t}\big(A(t)B(t)\big)=\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)B(t)+A(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}B(t)\right)\ . \tag{4.1.1}$$

Remark 4.1.5. Beware the order in which we multiply matrices in (4.1.1): it is *not* correct to invert the order of multiplication and write
$$\frac{\mathrm{d}}{\mathrm{d}t}\big(A(t)B(t)\big)=B(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)+\left(\frac{\mathrm{d}}{\mathrm{d}t}B(t)\right)A(t)\ ,$$
since matrix multiplication is not commutative. As a matter of fact, the products
$$B(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)\quad\text{and}\quad\left(\frac{\mathrm{d}}{\mathrm{d}t}B(t)\right)A(t)$$
might not even be well defined.

4.1.5. First-order linear systems. Having introduced matrix-valued functions and related notation, we may now write a general **linear system of n first-order differential equations in n unknown functions** $x_1(t),\dots,x_n(t)$,
$$\begin{cases} x_1'(t)=p_{11}(t)x_1(t)+p_{12}(t)x_2(t)+\cdots+p_{1n}(t)x_n(t)+f_1(t)\\ x_2'(t)=p_{21}(t)x_1(t)+p_{22}(t)x_2(t)+\cdots+p_{2n}(t)x_n(t)+f_2(t)\\ \cdots\\ x_n'(t)=p_{n1}(t)x_1(t)+p_{n2}(t)x_2(t)+\cdots+p_{nn}(t)x_n(t)+f_n(t)\end{cases}, \tag{4.1.2}$$
where $p_{ij}(t)$ is a continuous function for all $1\le i,j\le n$ and $f_i(t)$ is a continuous function for all $1\le i\le n$, in concise form as
$$x'(t)=P(t)x(t)+f(t)$$
where $x(t)$ is the column vector-valued function
$$x(t)=\begin{pmatrix}x_1(t)\\x_2(t)\\ \cdot\\ \cdot\\ x_n(t)\end{pmatrix},$$

$x'(t)$ is its derivative, $f(t)$ is the column vector-valued function

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \cdot \\ \cdot \\ f_n(t) \end{pmatrix} ,$$

and $P(t) = [p_{ij}(t)]$ is the matrix-valued function

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & \cdots & p_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & \cdots & p_{nn}(t) \end{pmatrix} ,$$

which we refer to as the **coefficient matrix** of the system.

A **solution** of the linear system (4.1.2), defined on an open interval $I \subset \mathbb{R}$, is a column vector-valued function $x(t)$ whose elements $x_1(t), \ldots, x_n(t)$ are continuously differentiable on I and satisfy the n differential equations of which the system (4.1.2) consists.

We would like to investigate solutions to a given linear system as in (4.1.2). The fundamental starting point, just as in the case of a single differential equation, is an abstract result on existence and uniqueness of solutions to initial value problems associated to linear systems of differential equations.

Theorem 4.1.6 (Existence and uniqueness of solutions to IVPs for linear systems). *Suppose the functions $p_{ij}(t)$, $1 \le i, j \le n$, and $f_i(t)$, $1 \le i \le n$ are all continuous on an open interval $I \subset \mathbb{R}$. Then, for every $a \in I$ and every column vector $b = \begin{pmatrix} b_1 & b_2 & \cdot & \cdot & b_n \end{pmatrix}^T$, there exists a unique solution $x(t) = \begin{pmatrix} x_1(t) & x_2(t) & \cdot & \cdot & x_n(t) \end{pmatrix}^T$ to the linear system (4.1.2) which is defined over the entire interval I and satisfies the initial condition $x(a) = b$.*

In order to understand the general structure of the set of solutions to the linear system (4.1.2) (that is, in order to understand its *general solution*), we first look at the associated **homogeneous system**

$$x'(t) = P(t)x(t) ,$$

which is obtained from (4.1.2) by replacing $f(t)$ with the zero column vector. For homogeneous linear systems, a superposition principle holds.

Theorem 4.1.7 (Principle of superposition). *Consider a homogeneous linear first-order system*

$$x'(t) = P(t)x(t) ,$$

and let $x^{(1)}(t), \ldots, x^{(n)}(t)$ be n vector-valued solutions of the system. If $c_1, \ldots, c_n$ are real numbers, then the linear combination

$$x(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t)$$

is also a solution of the system.

We provide the elementary proof of the principle.

Proof. Using the fact that differentiation of matrix-valued functions preserves addition and scalar multiplication, we can write

$$x'(t) = (c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t))' = c_1 x^{(1)}(t)' + \cdots + c_n x^{(n)}(t)' .$$

Now each $x^{(i)}(t)$ is a solution to the system, so that

$$x^{(i)}(t)' = P(t)x^{(i)}(t) \quad \text{for all } 1 \le i \le n .$$

We deduce from the above that

$$x'(t) = c_1 P(t) x^{(1)}(t) + \cdots + c_n P(t) x^{(n)}(t) = P(t)(c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t)) ,$$

where the last equality follows from distributivity of matrix multiplication with respect to matrix addition. We have thus shown that

$$x'(t) = P(t)x(t) ,$$

that is, $x(t)$ is a solution of the system, as desired. □

4.1.6. Independence and general solutions of homogeneous linear systems.

DEFINITION 4.1.8 (Linear dependence of functions). A collection of n column vector-valued functions $x^{(1)}(t), \ldots, x^{(n)}(t)$, defined over an open interval $I \subset \mathbb{R}$, is said to be **linearly dependent** if there exist real numbers $c_1, \ldots, c_n$, not all zero, such that

$$c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t) = 0 \quad \text{for all } t \in I.$$

If the functions are not linearly dependent, then they are called **linearly independent**.

DEFINITION 4.1.9 (Wronskian). Given n column vector-valued functions

$$x^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \cdot \\ \cdot \\ x_{n1}(t) \end{pmatrix} , \quad \cdots \quad , \quad x^{(n)}(t) = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \cdot \\ \cdot \\ x_{nn}(t) \end{pmatrix} ,$$

their **Wronskian** is the real-valued function given by the determinant

$$W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & \cdots & x_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & \cdots & x_{nn}(t) \end{vmatrix} .$$

PROPOSITION 4.1.10 (Wronskian of solutions). *Let $x^{(1)}(t), \ldots, x^{(n)}(t)$ be solutions to the homogeneous linear system*

$$x'(t) = P(t)x(t) ,$$

defined over an open interval $I \subset \mathbb{R}$. Then the following hold:

(1) if the functions $x^{(1)}(t), \ldots, x^{(n)}(t)$ are linearly dependent, then their Wronskian $W(t)$ vanishes for every $t \in I$.
(2) if the functions $x^{(1)}(t), \ldots, x^{(n)}(t)$ are linearly independent, then $W(t) \neq 0$ for all $t \in I$.

PROOF. The first assertion does not necessitate the assumption that the $x^{(i)}$'s are solutions to the given system; it is an elementary linear-algebraic consequence of linear dependence.

As far as the second assertion is concerned, suppose that there is $t_0 \in I$ such that $W(t_0) = 0$. It follows from linear algebra that the column vectors $x^{(1)}(t_0), \ldots, x^{(n)}(t_0)$ are linearly dependent: there are real numbers $c_1, \ldots, c_n$, not all zero, such that

$$c_1 x^{(1)}(t_0) + \cdots + c_n x^{(n)}(t_0) = 0 .$$

It follows that the column-vector valued function

$$x(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t)$$

is a solution to the given linear system (by the superposition principle) satisfying the initial condition $x(t_0) = 0$. Uniqueness of solutions to IVP's for linear systems yields that, necessarily,

$$x(t) = 0 ,$$

that is, the functions $x^{(1)}(t), \ldots, x^{(n)}(t)$ are linearly dependent. □

As for single differential equations, by the **general solution** of a linear system

$$x'(t) = P(t)x(t) + f(t)$$

we mean the set of all solutions of the system.

THEOREM 4.1.11 (General solution to homogeneous linear systems). *Consider a homogeneous linear system*

$$x'(t) = P(t)x(t) ,$$

and let $x^{(1)}(t), \dots, x^{(n)}(t)$ be linearly independent solutions of the system, defined over an open interval $I \subset \mathbb{R}$. If $x(t)$ is a solution of the system, then there exist real numbers $c_1, \dots, c_n$ such that

$$x(t) = c_1 x^{(1)}(t) + \cdots c_n x^{(n)}(t) .$$

As a consequence, the general solution of the system is given by

$$x(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t) , \quad c_1, \dots, c_n \in \mathbb{R}.$$

PROOF. Fix a point $t_0 \in I$, and let $b = x(t_0)$. Since the $x^{(i)}$'s are linearly independent, so are the vectors

$$b_1 = x^{(1)}(t_0), \dots, b_n = x^{(n)}(t_0) .$$

It follows from linear algebra that b can be expressed as a linear combination of the b_i's, namely

$$b = c_1 b_1 + \cdots + c_n b_n$$

for some real numbers $c_1, \dots, c_n$. Consider now the column vector-valued function

$$y(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t) ;$$

by the superposition principle, it is a solution to the given linear system. Furthermore, it satisfies the initial condition

$$y(t_0) = c_1 b_1 + \cdots c_n b_n = b$$

by construction. On the other hand, we already know that the function $x(t)$ is a solution to the very same IVP; uniqueness of solutions forces

$$y(t) = x(t) ,$$

that is, $x(t)$ is a linear combination of the $x^{(i)}$'s. □

4.1.7. General solutions of non-homogeneous linear systems.

THEOREM 4.1.12 (General solution to non-homogeneous linear systems). *Consider a non-homogeneous linear system*

$$x'(t) = P(t)x(t) + f(t) .$$

Let $x_p(t)$ be a particular solution of the system, and let $x^{(1)}(t), \dots, x^{(n)}(t)$ be linearly independent solutions of the associated homogeneous linear system

$$x'(t) = P(t)x(t) .$$

Then, the general solution of the non-homogeneous system is given by

$$x(t) = x_p(t) + c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t) , \quad c_1, \dots, c_n \in \mathbb{R}.$$

PROOF. Let $c_1, \dots, c_n$ be real numbers. Then, by the superposition principle, the linear combination

$$c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t)$$

is a solution to the homogeneous system

$$x'(t) = P(t)x(t) .$$

As a consequence, the function

$$x(t) = x_p(t) + c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t)$$

satisfies, by linearity of derivatives,

$$\begin{aligned} x'(t) &= x_p'(t) + (c_1x^{(1)}(t) + \cdots + c_nx^{(n)}(t))' = P(t)x_p(t) + f(t) + P(t)(c_1x^{(1)}(t) + \cdots + c_nx^{(n)}(t)) \\ &= P(t)x(t) + f(t) , \end{aligned}$$

that is, it is a solution to the given non-homogeneous linear system.

Conversely, suppose $x(t)$ solves the given non-homogeneous linear system. Then the difference $x(t) - x_p(t)$ satisfies

$$(x(t) - x_p(t))' = x(t)' - x_p(t)' = P(t)x(t) + f(t) - P(t)x_p(t) - f(t) = P(t)(x(t) - x_p(t)) ,$$

that is, it solves the associated homogeneous system. It follows that $x(t) - x_p(t)$ is a linear combination of the $x^{(i)}$'s, namely there are real numbers $c_1, \ldots, c_n$ such that

$$x(t) = x_p(t) + c_1x^{(1)}(t) + \cdots + c_nx^{(n)}(t) .$$

□

4.2. The eigenvalue method for homogeneous systems

4.2.1. First case: two distinct real eigenvalues.

Theorem 4.2.1. *Let*

$$x'(t) = Ax(t)$$

be a homogeneous 2×2 *linear system. Suppose that* A *has two distinct real eigenvalues* λ_1, λ_2. *Let* v_1 *and* v_2 *be eigenvectors of* A *relative to the eigenvalues* λ_1 *and* λ_2*, respectively. Then* v_1 *and* v_2 *are linearly independent. As a consequence, the the functions*

$$x^{(1)}(t) = e^{\lambda_1 t}v_1 , \quad x^{(2)}(t) = e^{\lambda_2 t}v_2$$

are linearly independent solutions to the given system, whose general solution is therefore given by

$$x(t) = c_1x^{(1)}(t) + c_2x^{(2)}(t) , \quad c_1, c_2 \in \mathbb{R}.$$

Proof. The fact that eigenvectors relative to distinct eigenvalues are linearly independent is standard result in linear algebra, and will not be demonstrated here.

Let us show that linear independence of v_1 and v_2 yields linear independence of $x^{(1)}$ and $x^{(2)}$. For that, we simply compute the Wronskian of $x^{(1)}$ and $x^{(2)}$ at $t = 0$:

$$W(0) = \begin{vmatrix} e^{\lambda_1 \cdot 0}v_1 & e^{\lambda_2 \cdot 0}v_2 \end{vmatrix} = \begin{vmatrix} v_1 & v_2 \end{vmatrix} \neq 0 ,$$

the last step following because the determinant of a matrix whose columns are linearly independent is non-zero. By Theorem **??**, we conclude that $x^{(1)}$ and $x^{(2)}$ are linearly independent column vector-valued functions.

The last assertion of the theorem follows directly from Theorem **??**. □

Example 4.2.2. Let us find a general solution of the homogeneous linear system

$$\begin{cases} x_1' = 4x_1 + 2x_2 \\ x_2' = 3x_1 - x_2 \end{cases} .$$

The coefficient matrix of the system is

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} .$$

We start by finding the eigenvalues of A: these are the roots of the polynomial

$$\det(A - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 6 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) .$$

We thus have two distinct real eigenvalues: $\lambda = -2$ and $\lambda = 5$.

We start with $\lambda = -2$, and find an associated eigenvector. This is a vector $v = (a, b)^T$ satisfying $(A + 2\mathbf{I})v = 0$, that is, the linear system

$$\begin{cases} 6a + 2b = 0 \\ 3a + b = 0 \end{cases} .$$

It is clear by inspection that we can choose $a = 1$, $b = -3$. We thus get a corresponding solution to our linear system of differential equations, given by

$$x^{(1)}(t) = e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ -3e^{-2t} \end{pmatrix} .$$

4.2.2. Second case: a pair of complex-conjugate eigenvalues. Suppose now we are in a situation were the characteristic polynomial of the 2×2 matrix A has no real roots. In this case, necessarily, it has two non-real complex-conjugate roots, which means that A has a pair of non-real complex conjugate eigenvalues,

$$\lambda_1 = \lambda , \quad \lambda_2 = \overline{\lambda} .$$

Let now v be an eigenvector of A corresponding to the eigenvalue λ. Taking the complex conjugate of the equation

$$Av = \lambda v ,$$

which is verified by definition of v, we obtain

$$\overline{A}\overline{v} = \overline{\lambda}\overline{v} ,$$

but since A has real coefficients, $\overline{A} = A$, whence we get

$$A\overline{v} = \overline{\lambda}\overline{v} .$$

It follows that the complex-conjugate vector $\overline{v}$ is an eigenvalue associated to the complex-conjugate eigenvalue $\overline{\lambda}$.

The general complex-valued solution to the homogeneous system

$$x'(t) = Ax$$

is thus given by

$$y(t) = c_1 e^{\lambda t} v + c_2 e^{\overline{\lambda} t} \overline{v} , \quad c_1, c_2 \in \mathbb{C}.$$

In order to get a pair of linearly independent real-valued solutions to the system, which is our ultimate goal, we resort to the superposition principle, which is equally valid for complex-valued solutions. The principle tells us that the *real* as well as the *imaginary* part of any complex-valued solution are real-valued solutions to the system: indeed, whenever $x(t)$ is a complex-valued solution to the system, we can write its real and imaginary parts, respectively, as

$$\operatorname{Re} y(t) = \frac{y(t) + \overline{y(t)}}{2} , \quad \operatorname{Im} y(t) = \frac{y(t) - \overline{y(t)}}{2i} .$$

Since the complex-conjugate $\overline{x(t)}$ is also a complex-valued solution to the system, as emerges from taking the complex-conjugate of the equation

$$y'(t) = Ay(t) ,$$

recalling that $\overline{A} = A$, the superposition principle yields that

$$x^{(1)}(t) = \operatorname{Re} y(t) , \quad x^{(2)}(t) = \operatorname{Im} y(t)$$

are real-valued solutions to the system. It is now a fact that, if $y(t)$ takes the form

$$y(t) = e^{\lambda t} v$$

for some non-real complex scalar λ and some non-real complex vector v, as above, then the resulting $x^{(1)}$ and $x^{(2)}$ are linearly independent. The general solution to the homogeneous system will thus be given by

$$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)\ , \quad c_1, c_2 \in \mathbb{R}.$$

THEOREM 4.2.3. *Let*

$$x'(t) = Ax(t)$$

be a homogeneous 2×2 *linear system. Suppose that* A *has a pair of non-real complex-conjugate eigenvalues* $\lambda_1 = \lambda, \lambda_2 = \overline{\lambda}$. *Let* v *be a complex eigenvector of* A *relative to the eigenvalue* λ. *If*

$$y(t) = e^{\lambda t} v$$

is the corresponding complex-valued solution to the system, then the functions

$$x^{(1)}(t) = \mathrm{Re}\, y(t)\ , \quad x^{(2)}(t) = \mathrm{Im}\, y(t)$$

are linearly independent real-valued solutions to the given system, whose general solution is therefore given by

$$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)\ , \quad c_1, c_2 \in \mathbb{R}.$$

EXAMPLE 4.2.4. Let's find the general solution to the linear system

$$\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 3x_1 + 4x_2 \end{cases} ,$$

The coefficient matrix is

$$A = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} ,$$

and its characteristic polynomial is

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = \lambda^2 - 8\lambda + 25\ .$$

We have thus a pair of complex-conjugate eigenvalues, namely

$$\lambda_{1,2} = 4 \pm \sqrt{-9} = 4 \pm 3i\ .$$

Let's find a complex eigenvector for $\lambda_1 = 4 + 3i$. We need to find a non-trivial complex solution to the linear system

$$\begin{cases} 4 - (4+3i)a - 3b = 0 \\ 3a + (4 - (4+3i))b = 0 \end{cases} ,$$

that is, to

$$\begin{cases} -3ia - 3b = 0 \\ 3a - 3ib = 0 \end{cases} .$$

Clearly, the first equation is simply the second one multiplied by $-i$, hence we are only left with the second equation

$$3a - 3ib = 0\ ;$$

a non-trivial solution for it is given by $a = i$, $b = 1$. Thus, the column vector

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

is an eigenvector for the eigenvalue $4 + 3i$. It follows that the complex vector-valued function

$$y(t) = e^{(4+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} ie^{(4+3i)t} \\ e^{(4+3i)t} \end{pmatrix}$$

is a complex-valued solution to our linear system.

In order to find real-valued solutions, we write

$$y(t) = e^{4t}\begin{pmatrix} ie^{3it} \\ e^{3it} \end{pmatrix} = e^{4t}\left(\begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + i\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}\right)$$

separating the real and imaginary parts. The latter are given, respectively, by

$$x^{(1)}(t) = e^{4t}\begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}, \quad x^{(2)}(t) = e^{4t}\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix},$$

and by the discussion above they are linearly independent real-valued solutions to our original system. The general solution is thus given by

$$x(t) = e^{4t}\left(c_1\begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + c_2\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}\right), \quad c_1, c_2 \in \mathbb{R}.$$

4.3. Multiple eigenvalue solutions

We finally deal with the last remaining case, when the characteristic polynomial of the coefficients matrix A has a double root, namely a single real root λ with multiplicity two. In this case, it is definitely possible, by definition of eigenvalue, to find an eigenvector v corresponding to the eigenvalue λ. Attached to it there is a solution

$$y(t) = e^{\lambda t}v$$

to our homogeneous system

$$x'(t) = Ax(t)\ .$$

However, it might not be possible to find two linearly independent solutions of the above form, or equivalently, it is not always the case that one can find two linearly independent eigenvectors v_1, v_2 relative to the single eigenvalue λ.

Example 4.3.1. Consider the homogeneous system

$$\begin{cases} x_1' = x_1 - x_2 \\ x_2' = x_1 + 3x_2 \end{cases}.$$

Its coefficient matrix is

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix},$$

whose characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2\ ;$$

we thus have a unique real root $\lambda = 2$, with double multiplicity. Let's determine all eigenvectors of A corresponding to its unique eigenvalue 2. We need to solve the linear system

$$\begin{cases} -a - b = 0 \\ a + b = 0 \end{cases};$$

the first equation is obtained by the second one multiplying both sides by -1, whence we are only left with the single equation

$$a + b = 0\ .$$

Solutions to this equation are given by

$$a = \alpha\ , \quad b = -\alpha$$

for every fixed real value of α; thus, the sought after eigenvectors are all scalar multiples of the vector

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

It is therefore impossible to find two linearly independent eigenvectors in this case.

Remark 4.3.2. It is a fact from linear algebra, which we shall not need, that when A has a single eigenvalue of double multiplicity, then it is possible to find linearly independent eigenvectors if and only if A takes the diagonal form

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$, thus if and only if the original homogeneous system is of the form

$$\begin{cases} x_1' = \lambda x_1 \\ x_2' = \lambda x_2 \end{cases} ; \tag{4.3.1}$$

in this case λ is the unique eigenvalue of A and *every* vector

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

is an eigenvector.

Observe that the homogeneous system in (4.3.1) can be treated as a pair of independent differential equations, one for the unknown x_1 and the other for the unknown x_2; there would be no need to develop the theory of homogeneous linear systems presented in this chapter to deal with trivial systems of this form.

When we treated homogeneous linear differential equations of the second order, in the case of a double root λ for the characteristic equation we found out that the pair

$$e^{\lambda t} , \quad te^{\lambda t}$$

consists of linearly independent solutions to the equation. This suggests we might try a similar approach for linear systems. Let us thus verify whether the column vector-valued function

$$te^{\lambda t} v ,$$

where λ is the unique eigenvalue of A and v is a choice of eigenvector for λ, is a solution to the system. We compute, using the product rule for derivatives of scalar multiplication,

$$(te^{\lambda t} v)' = (e^{\lambda t} + \lambda te^{\lambda t})v + te^{\lambda}tv' = e^{\lambda t}(1 + \lambda t)v ;$$

on the other hand we have that

$$A(te^{\lambda t} v) = te^{\lambda t} Av = te^{\lambda t} \lambda v = \lambda te^{\lambda t} v ,$$

whence the two last displayed expressions are equal, as functions of t, if and only if

$$\lambda t = 1 + \lambda t \quad \text{for all } t ,$$

which is clearly impossible.

We have thus verified that our initial guess $te^{\lambda t} v$ is *never* a solution to the system. However, we don't completely give up hope on the approach, and slightly generalize the type of functions we would like to consider as candidates for solutions to the system which are linearly independent from $e^{\lambda} tv$. We look for functions of the form

$$e^{\lambda t}(tv_1 + v_2)$$

where v_1 is an eigenvector of A corresponding to λ and v_2 is a second vector to be determined. In order to check whether the last displayed function is a solution, we compute

$$(e^{\lambda t}(tv_1 + v_2))' = \lambda e^{\lambda t}(tv_1 + v_2) + e^{\lambda t} v_1 = e^{\lambda t}((1 + \lambda t)v_1 + \lambda v_2) .$$

On the other hand, we have

$$A(e^{\lambda t}(tv_1 + v_2)) = e^{\lambda t}(tAv_1 + Av_2) = e^{\lambda t}(\lambda tv_1 + Av_2) ,$$

so that the function under consideration is a solution if and only if

$$(1 + \lambda t)v_1 + \lambda v_2 = \lambda tv_1 + Av_2$$

for all t, namely if and only if

$$v_1 = (A - \lambda)v_2 \ .$$

We thus need to look for column vectors v such that the vector

$$(A - \lambda\mathbf{I})v$$

is an eigenvector of A relative to the eigenvalue λ. In particular, if this is the case then

$$(A - \lambda\mathbf{I})^2 v = (A - \lambda\mathbf{I})(A - \lambda\mathbf{I})v = 0 \ .$$

Thus we would find such vectors v by finding non-zero solutions to the linear system

$$(A - \lambda\mathbf{I})^2 v = 0 \ ,$$

in the unknown $v = \begin{pmatrix} a & b \end{pmatrix}^T$ and with coefficient matrix $(A - \lambda\mathbf{I})^2$, and hoping that the vector

$$(A - \lambda\mathbf{I})v$$

is non-zero, in which case it is automatically an eigenvector of A relative to λ, as desired. To summarize, we need to find non-zero vectors v such that

$$(A - \lambda\mathbf{I})^2 v = 0 \quad \text{but} \quad (A - \lambda\mathbf{I})v \neq 0 \ .$$

If we are able to find such a vector v, then have reached our goal of find the general solution to our initial homogeneous system.

THEOREM 4.3.3. *Let*

$$x'(t) = Ax(t)$$

be a homogeneous 2×2 *linear system. Suppose that* A *has unique eigenvalue* λ*, and is not of the form*

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \ .$$

Then there is a vector v_2 *with the property that*

$$v_1 = (A - \lambda\mathbf{I})v_2 \neq 0 \quad \textit{and} \quad (A - \lambda\mathbf{I})^2 v_2 = (A - \lambda\mathbf{I})v_1 = 0 \ .$$

Furthermore, the functions

$$x^{(1)}(t) = e^{\lambda t} v_1 \ , \quad x^{(2)}(t) = e^{\lambda t}(t v_1 + v_2)$$

are linearly independent solutions of the given system, whose general solution is therefore given by

$$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) \ , \quad c_1, c_2 \in \mathbb{R}.$$

PROOF. The existence of a vector v_2 with the claimed property is a fact from linear algebra, for which we do not provide an argument here. The fact that $x^{(2)}(t)$ is a solution follows from the claimed properties of v_1 and v_2 and the discussion preceding the theorem. That $x^{(1)}(t)$ is a solution we already knew from previous sections, since v_1 is an eigenvector of A associated to the eigenvalue λ. It remains to show that $x^{(1)}$ and $x^{(2)}$ are linearly independent. For this, we evaluate their Wronskian at $t = 0$:

$$W(0) = \begin{vmatrix} e^{\lambda \cdot 0} v_1 & e^{(\lambda \cdot 0)}(0 v_1 + v_2) \end{vmatrix} = \begin{vmatrix} v_1 & v_2 \end{vmatrix} \ .$$

It is now a linear-algebraic fact that any pair (v_1, v_2) of vectors satisfying the claimed properties is linearly independent, whence $W(0) \neq 0$ and $x^{(1)}, x^{(2)}$ are linearly independent functions. □

EXAMPLE 4.3.4. We continue Example **??**. We found a unique eigenvalue $\lambda = 2$, all of whose eigenvectors are given by non-zero scalar multiples of the vector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \ .$$

According to Theorem 4.3.3, we need to find a vector v_2 with the property that

$$(A - \lambda\mathbf{I})^2 v_2 = 0 \quad \text{but} \quad v_1 = (A - \lambda\mathbf{I})v_2 \neq 0 \ .$$

We need to compute the matrix

$$(A-2\mathbf{I})^2 = (A-2\mathbf{I})(A-2\mathbf{I}) = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

Thus $(A-2\mathbf{I})^2$ is the zero matrix, so that *any* vector v_2 satisfies $(A-2\mathbf{I})^2 v_2 = 0$. We only need to make sure that $v_1 = (A-2\mathbf{I})v_2 \neq 0$, that is, that v_2 is not an eigenvector of A for its unique eigenvalue 2. For this, it is necessary and sufficient that v_2 is not a scalar multiple of $\begin{pmatrix} 1 & -1 \end{pmatrix}^T$, so that we can for instance choose

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad v_1 = (A-2\mathbf{I})v_2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} .$$

We conclude that the functions

$$x^{(1)}(t) = e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} , \quad x^{(2)}(t) = e^{2t} \left(t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

are linearly indepedent solutions to the system

$$\begin{cases} x_1' = x_1 - x_2 \\ x_2' = x_1 + 3x_2 \end{cases} ,$$

whose general solution is therefore given by

$$x(t) = e^{\lambda t} \left(c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left(t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right) , \quad c_1, c_2 \in \mathbb{R}.$$

In the previous example, we saw that the matrix

$$(A - \lambda \mathbf{I})^2 ,$$

where λ is the unique eigenvalue of A, is the zero matrix, so that we could pick any non-zero v_2 with the sole *caveat* that $v_1 = (A - \lambda\mathbf{I})v_2 \neq 0$. This is actually a general fact coming from linear algebra, which we will not use nor prove.

THEOREM 4.3.5. *Let A be a 2×2 matrix with a unique eigenvalue λ. Then the matrix*

$$(A - \lambda \mathbf{I})^2$$

is the zero matrix.

CHAPTER 5

Laplace transform methods

Functions in this chapter are real-valued.

5.1. Laplace transforms and inverse transforms

Definition 5.1.1 (Laplace transform). Let $f(t)$ be a function defined for all $t \geq 0$. The **Laplace transform** of f is the function F given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \ \mathrm{d}t$$

for all real values of s for which the improper integral is well defined and converges.

Recall that an improper integral over the infinite half-line $[a, \infty]$, $a \in \mathbb{R}$, is defined as a limit of integrals over bounded intervals:

$$\int_a^\infty g(t) \ \mathrm{d}t = \lim_{b \to \infty} \int_a^b g(t) \ \mathrm{d}t \ .$$

If the limit above exists, we say that the improper integral converges, else we say that it diverges.

Example 5.1.2. Consider the function $f(t) = 1$. Then the Laplace transform of $f(t)$, according to the definition, is given by

$$F(s) = \int_0^\infty e^{-st} \cdot 1 \ \mathrm{d}t = \lim_{b \to \infty} \int_0^b e^{-st} \ \mathrm{d}t = -\frac{1}{s} \lim_{b \to \infty} e^{-st}|_{t=0}^{t=b} = -\frac{1}{s}(-1 + \lim_{b \to \infty} e^{-sb}) \ ,$$

where the third inequality is valid provided that $s \neq 0$. We see that, under this assumption, the improper integral converges if and only if $s > 0$, in which case

$$F(s) = \frac{1}{s} \ .$$

For $s = 0$, we have the improper integral

$$\int_0^\infty 1 \ \mathrm{d}t \ ,$$

which is clearly divergent. Thus the Laplace transform of $f(t) = 1$ is given by

$$F(s) = \frac{1}{s}$$

for all $s > 0$.

Example 5.1.3. Consider the function $f(t) = e^{at}$, for some fixed real number a. The Laplace transform is then given by

$$F(s) = \int_0^\infty e^{-st} e^{at} \ \mathrm{d}t = \int_0^\infty e^{-(s-a)t} \ \mathrm{d}t = \frac{1}{a-s} \lim_{b \to \infty} e^{-(s-a)t}|_{t=0}^{t=b} = \frac{1}{a-s}(-1 + \lim_{b \to \infty} e^{-(s-a)b}) \ ,$$

where the third equality holds provided $s \neq a$. In this case, we see that the improper integral converges if and only if $s - a > 0$, that is, if and only if $s > a$. For $s = a$ we get

$$\int_0^\infty 1 \ \mathrm{d}t \ ,$$

which is divergent. Thus the Laplace transform of $f(t) = e^{at}$ is given by

$$F(s) = \frac{1}{s-a}$$

for all $s > a$.

EXAMPLE 5.1.4. We now compute the Laplace transform of a power function $f(t) = t^a$, where a is a fixed real number. It is most conveniently expressed in terms of the **Gamma function** $\Gamma(x)$, which is defined for all $x > 0$ by the formula

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, \mathrm{d}t \; .$$

Observe that the condition $x > 0$ ensures that the integral

$$\int_0^1 e^{-t} t^{x-1} \, \mathrm{d}t \; ,$$

which is improper for $x < 1$, converges; on the other hand, the improper integral

$$\int_1^\infty e^{-t} t^{x-1} \, \mathrm{d}t$$

is always convergent, as the exponential factor e^{-t} decreases to zero much faster than any polynomial function of t can increase to infinity.

Observe that

$$\Gamma(1) = \int_0^\infty e^{-t} \, \mathrm{d}t = \mathcal{L}\{1\}(1) = \frac{1}{1} = 1 \; .$$

Also, integration by parts yields that, for all $x > 0$,

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, \mathrm{d}t = -e^{-t} t^x |_{t=0}^{t=\infty} + x \int_0^\infty e^{-t} t^{x-1} \, \mathrm{d}t = x\Gamma(x) \; .$$

Thus

$$\Gamma(2) = 1\Gamma(1) = 1 \; , \quad \Gamma(3) = 2\Gamma(2) = 2 \; , \quad \Gamma(4) = 3\Gamma(3) = 3 \cdot 2 \; ,$$

and, inductively on the integer $n \geq 1$, we get that

$$\Gamma(n+1) = n! \; .$$

The Gamma function is thus naturally regarded as an extension of the factorial function to the real line.

As for the Laplace transform of t^a, we have

$$\mathcal{L}\{t^a\}(s) = \int_0^\infty e^{-st} t^a \, \mathrm{d}t = s^{-(a+1)} \int_0^\infty e^{-u} u^a \, \mathrm{d}u = s^{-(a+1)} \int_0^\infty e^{-u} u^{(a+1)-1} \, \mathrm{d}u = \frac{\Gamma(a+1)}{s^{a+1}}$$

where in the second equality we used the change of variable $u = st$. As the Gamma function is defined on the positive half-line, we deduce that

$$\mathcal{L}\{t^a\}(s) = \frac{\Gamma(a+1)}{s^{a+1}}$$

for all $s > 0$ and $a > -1$. In particular, for $n \geq 0$ an integer, we get

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$$

EXAMPLE 5.1.5. The identity

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$$

holds also for every complex number a with real part > -1. Since, for every $\omega \in \mathbb{R}$,

$$\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) \; ,$$

we get by linearity of the Laplace transform that

$$\mathcal{L}\{\cos(\omega t)\} = \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right) = \frac{s}{s^2+\omega^2} .$$

Similarly, since

$$\sin(\omega t) = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}) ,$$

we deduce that

$$\mathcal{L}\{\sin(\omega t)\} = \frac{1}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right) = \frac{\omega}{s^2+\omega^2} .$$

Alternatively, it is possible to derive the above expressions for the Laplace transform of $\cos(\omega t)$ and $\sin(\omega t)$ directly from the definition, by means of an integration by parts.

THEOREM 5.1.6 (Linearity of Laplace transforms). *Let $f(t), g(t)$ be two functions, a, b real numbers. Then*

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

for all s for which the Laplace transform of both f and g exists.

Define

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} .$$

EXAMPLE 5.1.7. Let $f(t) = u_a(t) = u(t-a)$. Its Laplace transform is given by

$$F(s) = \frac{1}{s} , \quad s > 0$$

if $a \leq 0$, as in this case u_a coincides with the function 1 over $[0, \infty)$, and by

$$F(s) = \int_0^\infty e^{-st} u_a(t) \ \mathrm{d}t = \int_a^\infty e^{-st} \ \mathrm{d}t = \frac{1}{s}(e^{-as} - \lim_{b\to\infty} e^{-sb}) = \frac{e^{-sa}}{s}$$

if $a > 0$.

THEOREM 5.1.8 (Existence of Laplace transforms). *Assume the $f(t)$ is piecewise continuous for $t \geq 0$ and satisfies the following: there exist real numbers $c, M, T > 0$ such that*

$$|f(t)| \leq Me^{ct}$$

for all $t \geq T$. Then the Laplace transform $\mathcal{L}\{f(t)\}$ exists for all $s > c$.

If $f(t)$ is a function satisfying the second assumption in the theorem, namely the existence of real numbers $c, M, T > 0$ such that

$$|f(t)| \leq Me^{ct}$$

for all $t \geq T$, we say that f is **of exponential order as** $t \to \infty$.

PROOF. Fix a real number $s > c$. We need to show that the improper integral

$$\int_0^\infty e^{-st} f(t) \ \mathrm{d}t$$

converges. Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any bounded interval, in particular the definite integral

$$\int_0^T e^{-st} f(t) \ \mathrm{d}t$$

exists and is finite. By additivity of integrals over disjoint intervals, it thus suffices to show that the improper integral

$$\int_T^\infty e^{-st} f(t) \ \mathrm{d}t$$

converges. Now, for any $b > T$, we have

$$\int_T^b |e^{-st} f(t)| \ \mathrm{d}t \le M \int_T^b e^{-(s-c)t} \ \mathrm{d}t = \frac{M}{s-c}(1 - e^{-(s-c)b}) \ ,$$

where the second inequality is a consequence of the exponential order assumption on $f(t)$. It follows that

$$\lim_{b\to\infty} \int_T^b |e^{-st} f(t)| \ \mathrm{d}t \le \frac{M}{s-c} \lim_{b\to\infty} (1 - e^{-(s-c)b}) = \frac{M}{s-c} \ ,$$

the last equality following from our assumption $s > c$. As a consequence, the improper integral

$$\int_T^\infty |e^{-st} f(t)| \ \mathrm{d}t$$

converges, and thus so does the improper integral

$$\int_T^\infty e^{-st} f(t) \ \mathrm{d}t \ ,$$

as claimed. □

Notice that, over the course of the previous proof, we have also shown that

$$\lim_{s\to+\infty} \int_T^\infty e^{-st} f(t) \, dt = 0 \ ;$$

observe also that, since $f(t)$ is uniformly bounded on the closed interval $[0, T]$ (by piecewise continuity), that is, there is $M' > 0$ such that

$$|f(t)| \le M'$$

for all $0 \le t \le M'$, we also have that

$$\lim_{s\to+\infty} \int_0^T e^{-st} f(t) \ \mathrm{d}t = 0 \ .$$

Combining the previous two limits, we deduce the following assertion.

Corollary 5.1.9. *Let the assumptions be as in Theorem 5.1.8. If $F(s)$ is the Laplace transform of $f(t)$, then*

$$\lim_{s\to\infty} F(s) = 0 \ .$$

Theorem 5.1.10 (Uniqueness of Laplace transforms). *Suppose $f(t)$ and $g(t)$ are piecewise continuous functions defined for $t \ge 0$, both of exponential order as $t \to \infty$. Assume their Laplace transform $F(s), G(s)$ satisfy*

$$F(s) = G(s) \quad \text{for all } s > c$$

for some $c > 0$. Then

$$f(t) = g(t)$$

for all $t \ge 0$ which is a point of continuity both for f and g.

Thus two piecewise continuous functions of exponential order having the same Laplace transform can differ only at their discontinuity points. In practice, this is not a serious problem since we shall apply the Laplace transform to solutions of differential equations, which are always everywhere continuous.

If $f(t)$ is a continuous function of exponential order defined for $t \ge 0$, then its Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is uniquely determined by f, namely there is no other continuous $g(t)$ of exponential order satisfying $\mathcal{L}\{g(t)\} = \mathcal{L}\{f(t)\}$. We can thus speak of $f(t)$ as the **inverse Laplace transform** of the function $F(s)$, and write

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \ .$$

EXAMPLE 5.1.11. We have seen in Example **??** that the Laplace transform of the function e^{at}, $a \in \mathbb{R}$, is $\frac{1}{s-a}$. This means, by definition, that e^{at} is the inverse transform of $\frac{1}{s-a}$:

$$e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}(t) \; .$$

EXAMPLE 5.1.12. In Example **??** we computed the Laplace transform of the function t^a, $a > -1$, which equals $\Gamma(a+1)/s^{a+1}$. Hence, using linearity of derivatives, we deduce that, for all $b > 00$,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^b}\right\}(t) = \frac{1}{\Gamma(b)}\mathcal{L}^{-1}\left\{\frac{\Gamma(b)}{s^b}\right\}(t) = \frac{t^{b-1}}{\Gamma(b)} \; .$$

5.2. Transformation of initial value problems

THEOREM 5.2.1 (Transform of derivatives). *Let $f(t)$ be a differentiable function, defined for $t \geq 0$, with piecewise continuous derivative $f'(t)$. Suppose f is of exponential order, namely there are $c, M, T > 0$ with*

$$|f(t)| \leq Me^{ct}$$

for all $t \geq T$. Then, for every $s > c$, the Laplace transform $\mathcal{L}\{f'(t)\}(s)$ is defined and satisfies

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}s - f(0) \; .$$

PROOF. The statement follows from a simple integration by parts:

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st}f'(t) \;\; \mathrm{d}t = e^{-st}f(t)|_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}f(t) \;\; \mathrm{d}t = s\mathcal{L}\{f(t)\}(s) - f(0) \; ,$$

where

$$\lim_{t\to\infty} e^{-st}f(t) = 0$$

holds in light of the exponential order assumption on f, combined with the fact that $s > c$. ☐

Consider an initial value problem associated to a second-order linear differential equation with constant coefficients:

$$ax''(t) + bx'(t) + cx(t) = f(t) \; , \quad x(0) = x_0 \; , \quad x'(0) = x_0' \tag{5.2.1}$$

where a, b, c, x_0, x_0' are real numbers and $f(t)$ is a given continuous function. Suppose $x(t)$ is a solution to the IVP which is of exponential order, and let $X(s)$ denote its Laplace transform, which is well defined for all sufficiently large values of s. Similarly, by virtue of Theorem 5.2.1, the Laplace transforms of $x'(t)$ and $x''(t)$ are well defined for all such values of s, and satisfy

$$\mathcal{L}\{x'\}(s) = sX(s) - x(0) = sX(s) - x_0 \; , \quad \mathcal{L}\{x''\}(s) = s\mathcal{L}\{x'\}(s) - x'(0) = s(sX(s) - x_0)) - x_0' \; .$$

Let also $F(s)$ be the Laplace transform of the function $f(t)$. Then taking Laplace transforms on both sides of the differential equation in (5.2.1), using linearity of Laplace transforms and the computations above, yields

$$a(s^2X(s) - x_0s - x_0') + b(sX(s) - x_0) + cX(s) = F(s) \; .$$

This is a purely algebraic (in particular, not differential) equation for the Laplace transform $X(s)$. Rearranging terms on the left-hand side, and transferring every summand everything not involing $X(s)$ to the right-hand side, we obtain

$$X(s)(as^2 + bs + c) = F(s) + ax_0s + ax_0' - bx_0 \; ,$$

whence, at least for those s such that $as^2 + bs + c \neq 0$ (there are at most two such s in $\mathbb{R}$), we get the Laplace transform $X(s)$ of the solution $x(t)$ satisfies

$$X(s) = \frac{F(s) + ax_0s + ax_0' - bx_0}{as^2 + bs + c} \; .$$

Since $x(t)$ is assumed to be of exponential order, Theorem 5.1.10 ensures the possibility of recovering $x(t)$ from its Laplace transform $X(s)$ by taking the inverse Laplace transform:

$$x(t) = \mathcal{L}^{-1}\{X(s)\}(t) \ .$$

EXAMPLE 5.2.2. Let's solve the initial value problem

$$x'' - x' - 6x = 0 \ , \quad x(0) = 2 \ , \quad x'(0) = -1$$

via the Laplace transform method. Let $X(s)$ denote the Laplace transform of $x(t)$; then the Laplace transform of $x'(t)$ is $sX(s) - x(0) = sX(s) - 2$, and the Laplace transform of $x''(t)$ is $s(sX(s) - 2) - x'(0) = s^2X(s) - 2s + 1$. Taking the Laplace transform of the differential equation, and observing that the Laplace transform of the zero function is the zero function, we get

$$s^2X(s) - 2s + 1 - (sX(s) - 2) - 6X(s) = 0 \ ,$$

that is,

$$X(s)(s^2 - s - 6) = 2s - 3 \ ,$$

whence

$$X(s) = \frac{2s - 3}{s^2 - s - 6} \ .$$

In order to find the solution $x(t)$, we need to take the inverse Laplace transform of the function we obtained for $X(s)$. We have computed in Example **??** the inverse transform of functions of the form $\frac{1}{s-a}$, $a \in \mathbb{R}$; here we can boil matters down to such functions via a partial fraction decomposition, and using linearity of the inverse transform. We write

$$\frac{2s-3}{s^2 - s - 6} = \frac{2s-3}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3} \ ,$$

and we need to find A and B such the previous equality is satisfied. We have

$$\frac{A}{s+2} + \frac{B}{s-3} = \frac{(A+B)s + 2B - 3A}{(s+2)(s-3)} \ ,$$

so that we need to solve the linear system

$$\begin{cases} A + B = 2 \\ -3A + 2B = -3 \end{cases} \ ;$$

Plugging $B = 2 - A$ into the second equation we obtain

$$-3A + 2(2 - A) = -3 \implies A = \frac{7}{5} \implies B = \frac{3}{5} \ .$$

Thus

$$x(t) = \mathcal{L}^{-1}\left\{\frac{2s-3}{s^2 - s - 6}\right\} = \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = \frac{7}{5}e^{-2t} + \frac{3}{5}e^{3t} \ .$$

EXAMPLE 5.2.3. We consider a forced damped spring-mass dashpot system, described by the differential equation

$$mx''(t) + cx'(t) + kx(t) = A\cos(\omega t) + B\sin(\omega t)$$

for fixed $A, B \in \mathbb{R}$, and subject to the initial conditions

$$x(0) = x_0 \ , \quad x'(0) = x_0'$$

for some given $x_0, x_0' \in \mathbb{R}$. Recall that m is the mass of the object attached to the spring, c is the resistance constant, and k is the spring constant, $c, k, m > 0$.

Letting $X(s)$ denote the Laplace transform of $x(t)$, we deduce the algebraic equation for $X(s)$:

$$m(s(sX(s) - x_0) - x_0') + c(sX(s) - x_0) + kX(s) = A\frac{s}{s^2 + \omega^2} + B\frac{\omega}{s^2 + \omega^2} \ ,$$

whence

$$X(s) = \frac{As + B\omega}{(s^2+\omega^2)(ms^2+cs+k)} + \frac{mx_0 s + mx_0' + cx_0}{ms^2+cs+k} \ .$$

We see thus that $X(s)$ can be written as a sum of two terms, one of which only depends only on the totality of the forces imparted upon the system, while the second one depends on the initial conditions but not on the external force exerted on the system.

Suppose, for instance, we have $A = B = \omega = m = 1$, $x_0 = 0$, $x_0' = 3$, $c = 5$, $k = 4$. Then

$$X(s) = \frac{s+1}{(s^2+1)(s^2+5s+4)} + \frac{3}{s^2+5s+4} \ . \tag{5.2.2}$$

In order to apply the inverse Laplace transform to recover the solution $x(t)$, we apply the method of partial fraction decomposition. Since $s^2 + 5s + 4 = (s+1)(s+4)$, we first write

$$\frac{3}{s^2+5s+4} = \frac{\alpha}{s+1} + \frac{\beta}{s+4} = \frac{(\alpha+\beta)s + 4\alpha + \beta}{(s+1)(s+4)} \ ,$$

from which we immediately get $\alpha = 1$, $\beta = -1$. As far as the first summand on the right-hand side of (5.2.2) is concerned, we decompose it as

$$\frac{s+1}{(s^2+1)(s^2+5s+4)} = \frac{1}{(s^2+1)(s+4)} = \frac{\alpha s+\beta}{s^2+1} + \frac{\gamma}{s+4} = \frac{(\alpha+\gamma)s^2 + (4\alpha+\beta)s + 4\beta+\gamma}{(s^2+1)(s+4)} \ .$$

We need to solve the 3×3 linear system

$$\begin{cases} \alpha + \gamma = 0 \\ 4\alpha + \beta = 0 \\ 4\beta + \gamma = 1 \end{cases} \ ,$$

which readily yields

$$\alpha = -\frac{1}{17} \ , \quad \beta = \frac{4}{17} \ , \quad \gamma = \frac{1}{17} \ .$$

In conclusion, applying linearity of the inverse Laplace transform, we deduce that

$$\begin{aligned} x(t) &= -\frac{1}{17}\mathcal{L}^{-1}\{\frac{s}{s^2+1}\} + \frac{4}{17}\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} + \frac{1}{17}\mathcal{L}^{-1}\{\frac{1}{s+4}\} + \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{1}{s+4}\} \\ &= -\frac{1}{17}\cos t + \frac{4}{17}\sin t - \frac{16}{17}e^{-4t} + e^{-t} \ . \end{aligned}$$

5.2.1. Laplace transform of integrals.

THEOREM 5.2.4 (Laplace transform of integrals). *Let $f(t)$ be a continuous function, defined for $t \geq 0$. Suppose f is of exponential order, namely there are $c, M, T > 0$ with*

$$|f(t)| \leq Me^{ct}$$

for all $t \geq T$. Set

$$g(t) = \int_0^t f(\tau) \, \mathrm{d}\tau \ .$$

Then, for every $s > c$, the Laplace transform $\mathcal{L}\{g(t)\}(s)$ is defined and satisfies

$$\mathcal{L}\{g(t)\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}s \ .$$

PROOF. Applying integration by parts and the fact that $g'(t) = f(t)$ by the fundamental theorem of calculus, we compute

$$\mathcal{L}\{g(t)\}(s) = \int_0^\infty e^{-st}g(t) \ \mathrm{d}t = -\frac{1}{s}e^{-st}g(t)|_{t=0}^{t=\infty} + \frac{1}{s}\int_0^\infty e^{-st}g'(t) \ \mathrm{d}t = \frac{g(0)}{s} + \frac{1}{s}\mathcal{L}\{f(t)\}(s) \ ,$$

which gives the desired conclusion as $g(0) = 0$ from the definition of g. □

Phrasing the previous theorem in terms of the inverse transform, we have that if

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \ ,$$

then

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau) \ \mathrm{d}\tau \ .$$

5.3. Periodic and piecewise continuous input functions

5.3.1. Transforms of some piecewise continuous functions.

THEOREM 5.3.1 (Laplace transform of translated functions). *Let $f(t)$ be a piecewise continuous function, and suppose the Laplace transform $F(s)$ of $f(t)$ exists for all $s > c$. Then the Laplace transform of the function $u(t-a)f(t-a)$ exists for all $s > c + a$ and, for all such values of s,*

$$\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s) \ .$$

The theorem admits, as usual, an interpretation in terms of inverse Laplace transforms. If

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \ ,$$

then

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

for all real values of a. Therefore, if we are able to explicitly compute the inverse Laplace transform of a certain function $F(s)$, we also known how to explicitly compute the Laplace transform of the function $e^{-as}F(s)$.

EXAMPLE 5.3.2 (Discontinuous forcing). We examine a forced undamped spring-mass system for which the motion of the mass is described by the initial value problem

$$x''(t) + 4x(t) = f(t) \ , \quad x(0) = x'(0) = 0 \ ,$$

where the external force $f(t) = \cos 2t$ is applied to the mass from time $t = 0$ to time $t = 2\pi$, and at time $t = 2\pi$ such force is turned off abruptly, the mass being then allowed to continue its motion unimpeded.

Taking the Laplace transform on both sides of the differential equation, and factoring in the initial conditions, we get

$$s^2X(s) + 4X(s) = F(s)$$

where $X(s) = \mathcal{L}\{f(t)\}$ and $F(s) = \mathcal{L}\{f(t)\}$. It follows that

$$X(s) = \frac{F(s)}{s^2+4} \ .$$

Let us now compute $F(s)$. Recall that the Laplace transform of the function $\cos 2t$ is given by

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4} \ .$$

The external force imparted on the mass is given by

$$f(t) = \begin{cases} \cos 2t & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases} \ ;$$

we can write it concisely, by means of the functions $u(t - \cdot)$, as

$$f(t) = (1 - u(t-2\pi))\cos 2t \ .$$

Thus, applying Theorem 5.3.1 together with linearity of the Laplace transform, we deduce that

$$F(s) = \frac{s(1-e^{-2\pi s})}{s^2+4} \ .$$

Therefore, we obtain

$$X(s) = \frac{s(1-e^{-2\pi s})}{(s^2+4)^2} \ .$$

In order to find the inverse transform of $X(s)$, and recover thus the unique solution $x(t)$ to our initial value problem, we first decompose $X(s)$ as the sum

$$X(s) = \frac{s}{(s^2+4)^s} - e^{-2\pi s}\frac{s}{(s^2+4)^2} \ , \tag{5.3.1}$$

and realize it suffices to find the inverse transform of

$$\frac{s}{(s^2+4)^2} \ ,$$

for in the second summand in (5.3.1) we can then apply Theorem 5.3.1.

Observe now that

$$\begin{aligned}
\mathcal{L}\{t\sin(\omega t)\} &= \int_0^\infty e^{-st} t\sin(\omega t) \ \mathrm{d}t = \frac{1}{2i}\int_0^\infty te^{-st}(e^{i\omega t} - e^{-i\omega t}) \ \mathrm{d}t \\
&= \frac{1}{2i}\int_0^\infty te^{-(s-i\omega)t} \ \mathrm{d}t - \frac{1}{2i}\int_0^\infty te^{-(s+i\omega)t} \ \mathrm{d}t \\
&= \frac{1}{2i}(\mathcal{L}\{t\}(s-i\omega) - \mathcal{L}\{t\}(s+i\omega)) \\
&= \frac{1}{2i}\left(\frac{1}{(s-i\omega)^2} - \frac{1}{(s+i\omega)^2}\right) \\
&= \frac{2\omega s}{(s^2+\omega^2)^2} \ .
\end{aligned}$$

It follows that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}\, t\sin 2t \ ,$$

and we can finally conclude that

$$x(t) = \frac{1}{4}\ \sin 2t\Big(t - u(t-2\pi)(t-2\pi)\Big) \ .$$

5.3.2. Transforms of periodic functions. We say that a piecewise continuous function $f(t)$, defined for all $t \geq 0$, is **periodic with period** p, where $p > 0$ is a given real number, if

$$f(t+p) = f(t)$$

for all $t \geq 0$.

THEOREM 5.3.3 (Laplace transform of periodic functions). *Let $f(t)$ be a piecewise continuous function. Suppose that $f(t)$ is periodic with period p. Then the Laplace transform of $F(s)$ of $f(t)$ exists for all $s > 0$ and is given by*

$$F(s) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t) \ \mathrm{d}t \ .$$

PROOF.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t) \ \mathrm{d}t = \sum_{N\in\mathbb{N}}\int_{Np}^{(N+1)p} e^{-st} f(t) \ \mathrm{d}t = \sum_{N\in\mathbb{N}}\int_0^p e^{-s(u+Np)} f(u+Np) \ \mathrm{d}u \\
&= \int_0^p e^{-st} f(t) \ \mathrm{d}t \ \sum_{N\in\mathbb{N}} e^{-sNp} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t) \ \mathrm{d}t \ .
\end{aligned}$$

□

EXAMPLE 5.3.4. Consider a spring-mass-dashpot system with $m = 1$, $c = 4$, $k = 20$. The system is initially at rest, namely $x(0) = x'(0) = 0$, and the mass is acted upon beginning at time $t = 0$ by an external force $f(t)$ which is 2π-periodic and satisfies

$$f(t) = \begin{cases} 20 & \text{if } 0 \le t < \pi \\ -20 & \text{if } \pi \le t < 2\pi \end{cases} .$$

Let's find the motion $x(t)$ of the mass for all times $t \ge 0$. Applyijng the Laplace transform to the differential equation

$$x''(t) + 4x'(t) + 20x(t) = f(t)$$

yields

$$s(sX(s) - x(0)) - x'(0) + 4(sX(s) - x(0)) + 20X(s) = F(s) ,$$

where, as usual, $X(s) = \mathcal{L}\{x(t)\}$ and $F(s) = \mathcal{L}\{f(t)\}$. We thus get the algebraic equation

$$(s^2 + 4s + 20)X(s) = F(s) ,$$

which implies that

$$X(s) = \frac{F(s)}{s^2 + 4s + 20} .$$

Let us compute $F(s)$ using periodicity and Theorem 5.3.3. We have

$$\begin{aligned} F(s) &= \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t) \, dt = \frac{20}{1 - e^{-2\pi s}} \left(\int_0^{\pi} e^{-st} \, dt - \int_{\pi}^{2\pi} e^{-st} \, dt \right) \\ &= -\frac{20}{s(1 - e^{-2\pi s})} \left(e^{-\pi s} - 1 - (e^{-2\pi s} - e^{-\pi s}) \right) = \frac{20}{s(1 - e^{-2\pi s})} (e^{-\pi s} - 1)^2 = \frac{20(1 - e^{-\pi s})}{s(1 + e^{-\pi s})} . \end{aligned}$$

We deduce that

$$X(s) = \frac{1 - e^{-\pi s}}{1 + e^{-\pi s}} \frac{20}{s(s^2 + 4s + 20)} .$$

5.4. Impulses and Delta Functions

In this section, we would like to model input external forces acting as impulses on the system under consideration, namely with high intensity over a very tiny period of time. We can think of it as a truly instantaneous impulse.

What matters is the value of the integral

$$p = \int_a^b f(t) \, dt ,$$

and not exactly how $f(t)$ varies over the interval $[a, b]$. We call p the **impulse** of the force $f(t)$ over the interval $[a, b]$. A good approximation of an instantaneous force acting with intensity $p = 1$ (we normalize the impulse for the sake of illustration) at a single instant of time $a > 0$ is the function

$$\delta_{a,\varepsilon}(t) = \begin{cases} 1/\varepsilon & \text{if } a \le t \le a + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

for very small values of ε. Indeed, the impulse of the force $\delta_{a,\varepsilon}(t)$ over $[0, +\infty]$ is

$$\frac{1}{\varepsilon} \int_a^{a+\varepsilon} 1 \, dt = 1 .$$

We would like to take the limit as $\varepsilon \to 0$ of such a construction. Observe, however, that if we take the *pointwise limit* of $\delta_{a,\varepsilon}(t)$ as $\varepsilon \to 0$ we get the function

$$\delta_a(t) = \begin{cases} +\infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} ;$$

on the other hand, if we could interchange limits and integral signs, we would get

$$1 = \lim_{\varepsilon\to 0} \int_0^{+\infty} \delta_{a,\varepsilon}(t) \ \mathrm{d}t = \int_0^{+\infty} \lim_{\varepsilon\to 0} \delta_{a,\varepsilon}(t) \ \mathrm{d}t = \int_0^{+\infty} \delta_a(t) \ \mathrm{d}t \ .$$

There clearly cannot exist any function $\delta_a(t)$ with the two above properies: any function which is non-zero only at a single point has vanishing integral. Yet, the usefulness of a mathematical object having both such properties, in an appropriately defined sense, is plain in modelling phenomena of instantaneous acting forces.

5.4.1. The Dirac delta function as an operator acting on continuous functions. We now attempt to give a precise mathematical meaning to the object $\delta_a(t)$, which we call **Dirac delta function**, in accordance with historical tradition, even though as we shall see it is not a function of the real variable t in the usual sense.

In order to make sense of $\delta_a(t)$, we start by a crucial observation. A piecewise continuous function $f(t)$ on $[0, +\infty)$, which vanishes outside a given bounded sub-interval is *almost* completely determined by all the values

$$\int_0^{\infty} g(t)f(t) \ \mathrm{d}t$$

for $g(t)$ ranging over the family of continuous functions on $[0, \infty)$. The precise meaning of the previous assertion is that, if $f_1(t)$ and $f_2(t)$ are piecewise continuous functions on $[0, +\infty)$, both vanishing outside a given bounded interval, and if

$$\int_0^{\infty} g(t)f_1(t) \ \mathrm{d}t = \int_0^{\infty} g(t)f_2(t) \ \mathrm{d}t$$

for all continuous $g(t)$ defined for $t \geq 0$, then $f_1(t) = f_2(t)$ for all points t at which both f_1 and f_2 are continuous.

Let us now compute the limit of the expression

$$\int_0^{+\infty} g(t)\delta_{a,\varepsilon}(t) \ \mathrm{d}t$$

as $\varepsilon \to 0$, for $g(t)$ as above. From the definition of $\delta_{a,\varepsilon}$, we are dealing with

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_a^{a+\varepsilon} g(t) \ \mathrm{d}t \ .$$

In order to compute such a limit, let $G(t)$ be an antiderivative of $g(t)$. Then, by the fundamental theorem of calculus,

$$\int_a^{a+\varepsilon} g(t) \ \mathrm{d}t = G(a+\varepsilon) - G(a) \ ,$$

and so the limit in question becomes

$$\lim_{\varepsilon\to 0} \frac{G(a+\varepsilon) - G(a)}{\varepsilon} = G'(a) = g(a)$$

by the the definition of derivative and the fact that G is an antiderivative of g.

We are thus lead to define the mathematical object $\delta_a(t)$ as the operation which, to each continuous function $g(t)$ defined for $t \geq 0$, assigns the number

$$\int_0^{+\infty} g(t)\delta_a(t) \ \mathrm{d}t = g(a) \ .$$

By virtue of the definition, it follows that the *Laplace transform* of the Dirac delta function $\delta_a(t)$ is given by

$$\mathcal{L}\{\delta_a(t)\} = \int_0^{+\infty} e^{-st}\delta_a(t) \ \mathrm{d}t = e^{-as} \tag{5.4.1}$$

for all s.

5.4.2. Delta function inputs. Notice that, even though $\delta_a(t)$ is formally not a function on $[0, +\infty)$, it is clear from the discussion above that the equality

$$\delta_a(t) = \delta_0(t - a)$$

holds, when appropriately interpreted. We adopt the notation $\delta(t)$ for $\delta_0(t)$.

EXAMPLE 5.4.1. We examine a mechanical system described by the initial value prioblem

$$x''(t) + 4x'(t) - 5x(t) = \delta(t - \pi) + \delta(t - 2\pi) \ , \quad x(0) = 0 \ , \ x'(0) = 2 \ .$$

Applying the Laplace transform on both sides of the differential equation, we obtain

$$s(sX(s) - x(0)) - x'(0) + 4(sX(s) - x(0)) - 5X(s) = \mathcal{L}\{\delta_\pi(t)\} + \mathcal{L}\{\delta_{2\pi}(t)\} \ ,$$

which, using the initial conditions and (5.4.1), becomes

$$s^2X(s) - 2 + 4sX(s) - 5X(s) = e^{-\pi s} + e^{-2\pi s} \ ,$$

that is,

$$X(s) = \frac{e^{-\pi s} + e^{-2\pi s} + 2}{s^2 + 4s - 5} \ .$$

We apply partial fraction decomposition to get

$$\frac{1}{s^2 + 4s - 5} = \frac{1}{6}\left(\frac{1}{s - 1} - \frac{1}{s - 5}\right) \ ,$$

which yields

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4s - 5}\right\} = \frac{1}{6}(e^{-t} - e^{-5t}) \ .$$

We conclude that

$$x(t) = \frac{1}{3}(e^{-t} - e^{-5t}) + \frac{1}{6}\left(u(t - \pi)(e^{-(t-\pi)} - e^{-5(t-\pi)}) + u(t - 2\pi)(e^{-(t-2\pi)} - e^{-5(t-2\pi)})\right) \ .$$

CHAPTER 6

Fourier series methods and partial differential equations

6.1. Periodic functions and trigonometric series

Consider the differential equation

$$x''(t) + \omega_0^2 x(t) = f(t) \ .$$

We have seen in Chapter **??** how to find its general solution, by the method of undetermined coefficients, when $f(t)$ is a linear combination of trigonometric functions of the form

$$f(t) = A\cos\omega t + B\sin\omega t \ , \quad A, B \in \mathbb{R}. \tag{6.1.1}$$

Several physically relevant external forces are periodic functions, namely they satisfy

$$f(t+p) = f(t)$$

for some $p > 0$. The superposition principle for nonhomogeneous equations allows to treat cases where $f(t)$ is, more generally with respect to (6.1.1), a linear combination

$$\sum_{n=1}^{N} \big(a_n \cos n\omega t + b_n \sin n\omega t\big) \ .$$

The French mathematician Joseph Fourier, in his epoch-making treatise "Théorie Analytique de la Chaleur" (1822), asserted that *every* 2π-periodic function can be written as an infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos nt + b_n \sin nt\big) \ .$$

We have the formulas

$$\int_{-\pi}^{\pi} \cos mt \cos nt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \ ,$$

$$\int_{-\pi}^{\pi} \sin mt \sin nt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \ ,$$

$$\int_{-\pi}^{\pi} \cos mt \sin nt = 0 \ ,$$

for all integers $m, n \geq 0$.

Suppose

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos nt + b_n \sin nt\big) \ .$$

Then

$$\int_{-\pi}^{\pi} f(t) \ \mathrm{d}t = \pi a_0 + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nt + b_n \int_{-\pi}^{\pi} \sin nt \ \ \mathrm{d}t = \pi a_0 \ ,$$

whence

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \ \mathrm{d}t \ .$$

We proceed with

$$\int_{-\pi}^{\pi} f(t)\cos t \ \mathrm{d}t = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos t \ \mathrm{d}t + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nt\cos t \ \mathrm{d}t + b_n\int_{-\pi}^{\pi}\sin nt\cos t \ \mathrm{d}t = \pi a_1 \ ,$$

whence

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos t \ \mathrm{d}t \ .$$

Similarly

$$\int_{-\pi}^{\pi} f(t)\sin t \ \mathrm{d}t = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin t \ \mathrm{d}t + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nt\sin t \ \mathrm{d}t + b_n\int_{-\pi}^{\pi}\sin nt\sin t \ \mathrm{d}t = \pi b_1 \ ,$$

from which we deduce

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin t \ \mathrm{d}t \ .$$

Definition 6.1.1 (Fourier series of a piecewise continuous function). Let $f(t)$ be a piecewise continuous function, defined for all real values of t and periodic with period 2π. The **Fourier series** of $f(t)$ is defined as the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos nt + b_n\sin nt\big)$$

where the coefficients a_n and b_n are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt \ \mathrm{d}t$$

for $n \geq 0$ and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt \ \mathrm{d}t$$

for $n \geq 1$.

6.2. General Fourier series and convergence

6.3. Fourier sine and cosine series

Given a piecewise continuous function $f(t)$ defined for $0 < t < L$, where $L > 0$ is a fixed real number, we define its *even and odd extensions* as follows. The **even period $2L$ extension of f** is defined as the function

$$f_E(t) = \begin{cases} f(t) & \text{if } 0 < t < L \\ f(-t) & \text{if } -L < t < 0 \end{cases}$$

and by the $(2L)$-periodic condition

$$f(t+2L) = f(t)$$

for all other values of t. Similarly, the **odd period $2L$ extension of f** is defined as the function

$$f_O(t) = \begin{cases} f(t) & \text{if } 0 < t < L \\ -f(-t) & \text{if } -L < t < 0 \end{cases}$$

and by the $(2L)$-periodic condition

$$f(t+2L) = f(t)$$

for all other values of t.

REMARK 6.3.1. Notice that the requirements above identify the values of $f_E(t)$ and $f_O(t)$ uniquely for all real t which are not integer multiples of L. We adopt the convention of definining $f_E(t)$ and $f_O(t)$, at all such values of t, to be, respectively, the averages

$$\frac{f_E(t+) + f_E(t-)}{2}, \quad \frac{f_O(t+) + f_O(t-)}{2}$$

so that we ensure convergence of the Fourier series of f_E and f_O for all $t \in \mathbb{R}$, provided that $f(t)$ is piecewise smooth on $(0, L)$.

The Fourier series of the even extension f_E is the Fourier series of an even function, and as such takes the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right)$$

with

$$a_n = \frac{2}{L} \int_0^L f_E(t) \cos\left(\frac{n\pi t}{L}\right) \, \mathrm{d}t = \frac{2}{L} \int_0^L f(t) \cos\left(\frac{n\pi t}{L}\right) \, \mathrm{d}t \, .$$

Analogously, the Fourier series of the odd extension f_O is the Fourier series of an odd function, and as such takes the form

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

with

$$b_n = \frac{2}{L} \int_0^L f_O(t) \sin\left(\frac{n\pi t}{L}\right) \, \mathrm{d}t = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi t}{L}\right) \, \mathrm{d}t \, .$$

The previous two Fourier series are defined as the *Fourier cosine and Fourier sine series* of $f(t)$, respectively.

DEFINITION 6.3.2 (Fourier cosine and sine series). Let $f(t)$ be a piecewise continuous function defined for $0 < t < L$, where $L > 0$ is a given real number.

We define the **Fourier cosine series** of $f(t)$ as the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right)$$

where

$$a_n = \frac{2}{L} \int_0^L f(t) \cos\left(\frac{n\pi t}{L}\right) \, \mathrm{d}t$$

for all integers $n \geq 0$.

We define the **Fourier sine series** of $f(t)$ as the infinite series

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

where

$$b_n = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi t}{L}\right) \, \mathrm{d}t$$

for all integers $n \geq 0$.

COROLLARY 6.3.3. *Suppose $f(t)$ is a piecewise smooth function defined for $0 < t < L$, where $L > 0$ is a given real number. Then both the Fourier cosine series and the Fourier sine series of $f(t)$ converge to*

$$\frac{f(t+) + f(t-)}{2}$$

for all $0 < t < L$.

6.4. Heat conduction and separation of variables

The first prominent example of partial differential equation we shall investigate is the one-dimensional **heat equation**

$$u_t = ku_{xx} \tag{6.4.1}$$

where $k > 0$ is a fixed real number. It describes the temperature $u(x,t)$ of a heated rod at position $0 < x < L$, where $L > 0$ is a fixed real number representing the length of the rod, and at time $t > 0$. The constant $k > 0$ is known as the *thermal diffusivity* of the rod material. As in the context of ordinary differential equations, we associate further pieces of data to a partial differential equation which ensure uniqueness of solutions. In the realm of partial differential equations, such additional conditions are routinely presented in the form of **boundary value problems**. In the case of the heat equation, this amounts to specify the initial (namely, at time $t = 0$) temperature profile

$$u(x,0) = f(x)$$

of the rod, where we shall always assume that $f(x)$ is a piecewise smooth function defined for $0 < x < L$. Also, we specify some boundary conditions of the rod temperature at $x = 0$ and $x = L$, which are to be valid for all times $t \geq 0$. These come in two sorts:

(1) (zero endpoint temperatures) we impose

$$u(0,t) = u(L,t) = 0 \quad \text{for all } t \geq 0\ ;$$

(2) (insulated ends) we impose

$$u_x(0,t) = u_x(L,t) = 0 \quad \text{for all } t \geq 0\ .$$

In order to find solutions to our boundary value problems, we proceed via the method of separation of variables, which consists in looking for solutions of the form

$$u(x,t) = X(x)T(t) \tag{6.4.2}$$

where $X(x)$ is a twice-differentiable function of one real variable x and $T(t)$ is a differentiable function of one real variable t. Plugging (6.4.2) into (6.4.1), we get the equation

$$X(x)T'(t) = kX''(x)T(t)\ .$$

Assuming for the time being that $T(t) \neq 0$ and $X(x) \neq 0$, we can divide both sides of the last displayed equation by $X(x)T(t)$, and thus get

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)}\ .$$

It follows that there must exist a real number λ such that

$$\frac{T'(t)}{kT(t)} = -\lambda \quad \text{and} \quad \frac{X''(x)}{X(x)} = -\lambda\ ,$$

that is, the functions $X(x)$ and $T(t)$ must satisfy the ordinary differential equations

$$X''(x) + \lambda X(x) = 0$$

$$T'(t) + \lambda kT(t) = 0\ .$$

6.4.1. Zero endpoint temperatures. We start with the first equation. To begin with, notice that the function $X(x)$ must satisfy the endpoint value problem

$$X''(x) + \lambda X(x) = 0\ , \quad X(0) = X(L) = 0\ ; \tag{6.4.3}$$

indeed, the endpoint conditions for X are derived from the endpoints conditions

$$0 = u(0,t) = X(0)T(t) = X(L)T(t) = u(L,t)$$

for $u(x,t)$, since we are seeking non-trivial solutions $T(t) \neq 0$. Now (6.4.3) is an eigenvalue problem we have already discussed in §**??**, where we found that the problem admits a non-trivial (meaning, non identically vanishing) solution $X(x)$ if and only if λ is of the form

$$\lambda_n = \frac{\pi^2}{L^2}\, n^2 \ , \quad n \geq 1 \text{ an integer,} \tag{6.4.4}$$

with an associated eigenfunction of the form

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right) .$$

We now proceed with the consideration of the ODE for $T(t)$, with λ being one of the λ_n's found in (6.4.4). Either via separation of variables, or via the direct solution method for linear first-order ODEs, we see that a non-trivial solution to

$$T'(t) + \lambda_n k T(t) = 0$$

is given by

$$T_n(t) = e^{-\lambda_n k t} .$$

Hence, we have shown that the function

$$u_n(x,t) = X_n(x)T_n(t) = e^{-\pi^2 n^2 kt/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

solves the problem

$$u_t = k u_{xx} \ , \quad u(0,t) = u(L,t) = 0 \ .$$

It remains to ensure the initial condition

$$u(x,0) = f(x) \ ,$$

which is not satisfied by $u_n(x,t)$ unless we are in the very special case

$$f(x) = \sin\left(\frac{n\pi x}{L}\right) .$$

We now resort to the **principle of superposition of solutions**, which we already enunciated several times in the context of ordinary differential equations, and a version of which is also valid for linear partial differential equations such as the heat equation. We formulate it in the the specific context of the heat equation.

THEOREM 6.4.1 (Superposition principle). *Consider the boundary value problem*

$$u_t = k u_{xx} \ , \quad u(0,t) = u(L,t) = 0 \ , \quad u(x,0) = f(x) \tag{6.4.5}$$

where

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

for some functions $f_n(x)$ and some real numbers c_n, $n \geq 1$ an integer. Suppose, for every $n \geq 1$, that $u_n(x,t)$ is a solution to the boundary value problem

$$u_t = k u_{xx} \ , \quad u(0,t) = u(L,t) = 0 \ , \quad u(x,0) = f_n(x) \ .$$

Then the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

is a solution of the boundary value problem (6.4.5), provided it converges for all relevant pairs (x,t) and it satisfies the necessary regularity assumptions.

REMARK 6.4.2. An entirely analogous statement is valid for the case of insulated ends, where the condiiton

$$u(0,t) = u(L,t) = 0$$

is replaced by the condition

$$u_x(0,t) = u_x(L,t) = 0 \ .$$

In light of the previous theorem, in order to solve the BVP

$$u_t = k u_{xx} \ , \quad u(x,0) = f(x) \ , \quad u(0,t) = u(L,t) = 0 \ ,$$

we only need to decompose $f(x)$ according to its Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x$$

for all $n \geq 1$. From the previous discussion, in particular from the superposition principle, we then know that

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\pi^2 n^2 kt/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

is the sought after solution.

THEOREM 6.4.3 (Heat equation with zero endpoint temperature). *The unique solution to the boundary value problem*

$$u_t = k u_{xx} \ , \quad u(x,0) = f(x) \ , \quad u(0,t) = u(L,t)$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\pi^2 n^2 kt/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

is the Fourier sine series of $f(x)$.

EXAMPLE 6.4.4. Let's find the unique solution to the boundary value problem

$$u_t = 2u_{xx} \ , \quad u(x,0) = 5\sin(\pi x) - \frac{1}{5}\sin(3\pi x) \ , u(0,t) = u(1,t) = 0 \ .$$