Calculus with analytic geometry II

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Notation

If A and B are sets, the symbol $A \setminus B$ indicates the set of all elements of A which do not belong to B.

We indicate with $\mathbb{N}$ the set of natural numbers $\{0, 1, \dots, n, \dots\}$, with $\mathbb{Z}$ the set of rational integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$, and with $\mathbb{R}$ the set of real numbers.

CHAPTER 1

Integration techniques

This chapter concerns itself with various methods to evaluate definite and indefinite integrals, all of which serve the purpose of reducing an initial integrand to a more elementary one, whose primitive is hopefully known.

1.1. The substitution method

The chain rule for derivatives, recalled in Proposition B.2.2 of the Appendix, allows to compute the derivative of the composition of two differentiable functions. Since, by the fundamental theorem of calculus, integration can be regarded as an inverse operation with respect to differentiation, it stands to reason to expect an integral counterpart to the chain rule, namely a rule to integrate the composition of two integrable functions. Such a rule does indeed exists, and goes under the name of *substitution method*; indeed, as we shall see, it is implemented by substituting a new variable of integration, say u, in place of a convenient function of the original variable of integration x.

Theorem 1.1.1 (Substitution method, indefinite integral version). *Let $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$, and let $f\colon (a,b) \to \mathbb{R}$, $\varphi\colon (c,d) \to (a,b)$ be two functions. Assume that f admits an anti-derivative F on (a,b), and that φ is differentiable on (c,d). Then the function $F \circ \varphi$ is an anti-derivative of the product $(f \circ \varphi) \cdot \varphi'$ on (c,d). As a consequence,*

$$\begin{aligned}&\{G\colon (c,d) \to \mathbb{R} : G \textit{ is an anti-derivative of } (f\circ\varphi)\cdot\varphi' \textit{ on } (c,d)\} =\\ =&\{H \circ \varphi : H\colon (a,b) \to \mathbb{R} \textit{ is an anti-derivative of } f \textit{ on } (a,b)\} ;\end{aligned}$$

the previous equality of sets is written in indefinite-integral notation as

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = F(\varphi(x)) + C\,, \quad C \in \mathbb{R} \tag{1.1.1}$$

or, equivalently, as

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \left(\int f(u)\,\mathrm{d}u\right) \circ \varphi\,. \tag{1.1.2}$$

Proof. By the assumption, $F'(x) = f(x)$ for every $x \in (a,b)$. The chain rule for derivatives gives thus, for every $x \in (c,d)$

$$(F\circ\varphi)'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x)\,,$$

which precisely shows, by definition of anti-derivative, that $F\circ\varphi$ is an anti-derivative of $(f\circ\varphi)\cdot\varphi'$. On the other hand, as $F\circ\varphi$ is a given anti-derivative of $(f\circ\varphi)\cdot\varphi'$ on (c,d), we know by Proposition C.2.2 in the Appendix that all the others are given by

$$G(x) = F(\varphi(x)) + C\,, \quad x \in (c,d)$$

for C varying over the real numbers. We may clearly rewrite the last function as

$$G(x) = F(\varphi(x)) + C = (F + C)(\varphi(x))\,, \quad x \in (c,d)\,,$$

so that we deduce

$$\begin{aligned}&\{G\colon (c,d) \to \mathbb{R} : G \text{ is an anti-derivative of } (f\circ\varphi)\cdot\varphi' \text{ on } (c,d)\}\\ =&\{(F + C)\circ\varphi : F\colon (a,b) \to \mathbb{R} \text{ is a fixed anti-derivative of } f \text{ on } (a,b)\}\\ =&\{H \circ \varphi : H\colon (a,b) \to \mathbb{R} \text{ is an anti-derivative of } f \text{ on } (a,b)\}\,,\end{aligned}$$

as claimed. $\square$

The Fundamental Theorem of Calculus readily delivers the definite-integral version of the substitution method.

THEOREM 1.1.2 (Substitution method, definite integral version). *Let $a, b, c, c', d, d' \in \mathbb{R}$, $a < b$, $c' < c < d < d'$, and let $f\colon (a,b) \to \mathbb{R}$, $\varphi\colon (c', d') \to (a,b)$ be two functions. Assume that f is continuous on (a,b) and φ is continuously differentiable on (c', d'). Then*

$$\int_c^d f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int_{\varphi(c)}^{\varphi(d)} f(u)\,\mathrm{d}u\,. \tag{1.1.3}$$

PROOF. We shall derive it from Theorem 1.1.1. As f is continuous on (a,b), it admits a primitive F on it by the Fundamental Theorem of Calculus. For instance, we might take F to be

$$F(x) = \int_{t_0}^{x} f(t)\,\mathrm{d}t\,, \quad x \in (a,b)$$

where $t_0 \in (a,b)$ is a fixed base point. Therefore, $f\colon (a,b) \to \mathbb{R}$ and $\varphi\colon (c', d') \to (a,b)$ satisfy the assumptions of Theorem 1.1.1; we deduce that $F \circ \varphi$ is an anti-derivative of $(f \circ \varphi) \cdot \varphi'$ on (c,d). As $[c,d] \subset (c', d')$ by assumption, the latter function is continuously differentiable on $[c,d]$, as it is so on (c', d') by hypothesis. We may thus apply the Fundamental Theorem of Calculus to $(f \circ \varphi) \cdot \varphi'$ on $[c,d]$, which yields

$$\int_c^d f(\varphi(x))\varphi'(x)\,\mathrm{d}x = (F \circ \varphi)(d) - (F \circ \varphi)(c)\,. \tag{1.1.4}$$

Once more by the Fundamental Theorem of Calculus applied to the function f on the smallest closed interval containing $\varphi(c)$ and $\varphi(d)$, we see that

$$F(\varphi(d)) - F(\varphi(c)) = \int_{\varphi(c)}^{\varphi(d)} f(u)\,\mathrm{d}u$$

since F is an anti-derivative of f on (a,b). This concludes the proof. $\square$

REMARK 1.1.3. Observe that the statement of Theorem 1.1.2 does not assume that $\varphi(c) \leq \varphi(d)$; hence, it is equally valid in the case $\varphi(c) > \varphi(d)$, recalling that the definite integral on the right-hand side of (1.1.3) is defined as

$$\int_{\varphi(c)}^{\varphi(d)} f(u)\,\mathrm{d}u = -\int_{\varphi(d)}^{\varphi(c)} f(u)\,\mathrm{d}u\,.$$

Here is the standard way to memorize the change-of-variable formulas (1.1.2) and (1.1.3). We substitute[1] a new variable u in place of the function $\varphi(x)$, setting explicitly $u = \varphi(x)$. Now recall the notation $\frac{d\varphi}{\mathrm{d}x}$, introduced in Calculus I, for the derivative of the function u; as $u = \varphi(x)$, we may thus write $\frac{\mathrm{d}u}{\mathrm{d}x} = \varphi'(x)$; at this point, we formally[2] multiply both sides of the previous equality by $\mathrm{d}x$, thereby getting $\mathrm{d}u = \varphi'(x)\mathrm{d}x$. Finally, we also need to express the two boundaries of integration in terms of the new variable u: if c and d are the two boundaries in terms of x, then $\varphi(c)$ and $\varphi(d)$ are the new boundaries in terms of u, again since $u = \varphi(x)$. We see thus that the equality (1.1.3) is precisely obtained by setting in the integrand $u = \varphi(x)$, $\mathrm{d}u = \varphi'(x)\mathrm{d}x$, and replacing the boundaries of integration as indicated.

EXAMPLE 1.1.4. Let's compute

$$\int_1^2 x e^{x^2}\mathrm{d}x$$

[1] Hence the name of the method.

[2] For this course, the symbols $\mathrm{d}x$ and $\mathrm{d}u$ represent no mathematical quantity, hence such passage is logically unjustified. However, it represents a convenient way to memorize the rule to integrate by substitution.

using the substitution method. The function appearing inside the exponential, that is, x^2 has derivative $2x$, which apart from the constant factor 2 is precisely what the exponential gets multiplied by. To make the 2 appear inside the integral, we simply multiply and divide the original expression by 2. More precisely: we change variable by setting $u = x^2$, so that $\mathrm{d}u = 2x\mathrm{d}x$, the endpoints of the new interval of integration become $u(1) = 1^2 = 1$ and $u(2) = 2^2 = 4$, and thus we may write

$$\int_1^2 xe^{x^2}\,\mathrm{d}x = \frac{1}{2}\int_1^2 e^{x^2}(2x)\mathrm{d}x = \frac{1}{2}\int_1^4 e^u\,\mathrm{d}u = \frac{1}{2}\,e^u\Big|_1^4 = \frac{e^4-e}{2}$$

EXAMPLE 1.1.5. We now evaluate the indefinite integral

$$\int x^4\cos\left(x^5+2\right)\,\mathrm{d}x\ .$$

We realize that the derivative of x^5+2, which is the argument of the cosine function, is $5x^4$, that is, up to the constant factor 5, it's the factor multiplying the cosine. Hence it is sensible to use the substitution $u = x^5+2$, leading to $\mathrm{d}u = 5x^4\mathrm{d}x$, and thus to

$$\int x^4\cos\left(x^5+2\right)\,\mathrm{d}x = \frac{1}{5}\int\cos\left(x^5+2\right)(5x^4)\,\mathrm{d}x = \frac{1}{5}\int\cos u\,\mathrm{d}u = \frac{1}{5}\sin u + C\ ,\quad C\in\mathbb{R}.$$

As $u = x^5+2$, we deduce

$$\int x^4\cos\left(x^5+2\right)\,\mathrm{d}x = \frac{1}{5}\sin\left(x^5+2\right) + C\ ,\quad C\in\mathbb{R}.$$

EXAMPLE 1.1.6. Let us calculate

$$\int x^2\sqrt{x+3}\,\mathrm{d}x\ .$$

We apply the substitution $u = x+3$, which yields $\mathrm{d}u = \mathrm{d}x$; in order to express the factor x^2 in terms of the new variable u, we solve the equation $u = x+3$ for x, which results in $x = u-3$ and thus in $x^2 = (u-3)^2$. Therefore, we get

$$\begin{aligned}\int x^2\sqrt{x+3}\,\mathrm{d}x &= \int(u-3)^2\sqrt{u}\,\mathrm{d}u = \int u^{5/2}-6u^{3/2}+9u^{1/2}\,\mathrm{d}u = \frac{2}{7}u^{7/2}-\frac{12}{5}u^{5/2}+6u^{3/2}+C\\ &= \frac{2}{7}(x+3)^{7/2}-\frac{12}{5}(x+3)^{5/2}+6(x+3)^{3/2}+C\ ,\quad C\in\mathbb{R}.\end{aligned}$$

In Example 1.1.9, we shall see another possible substitution which helps solve this indefinite integral.

EXAMPLE 1.1.7. We now calculate

$$\int\frac{3x+2}{(x+1)^4}\,\mathrm{d}x\ .$$

Observe that it is easy to integrate the functions $\frac{1}{(x+1)^m}$, where $m\geq 1$ is an integer. The insight here is that the integrand above can be written as the difference of two such functions. Indeed, we have

$$\frac{3x+2}{(x+1)^4} = \frac{3x+3}{(x+1)^4}-\frac{1}{(x+1)^4} = \frac{3}{(x+1)^3}-\frac{1}{(x+1)^4}\ .$$

Now an elementary substitution $u = x+1$ in the two last displayed summands yields

$$\int\frac{3x+2}{(x+1)^4}\,\mathrm{d}x = -\frac{3}{2(x+1)^2}+\frac{1}{3(x+1)^3}+C\ ,\quad C\in\mathbb{R}.$$

EXAMPLE 1.1.8. We compute

$$\int\frac{x}{x^2+4}\,\mathrm{d}x\ ,\quad\int\frac{1}{x^2+4}\,\mathrm{d}x\ .$$

For the first one, we observe that the derivative of the denominator x^2+4 is $2x$, so that setting $u = x^2+4$, resulting in $\mathrm{d}u = 2x\mathrm{d}x$, we get

$$\int \frac{x}{x^2+4}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{x^2+4}\,2x\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\log|u| + C = \frac{1}{2}\log\left(x^2+4\right) + C\ , \quad C \in \mathbb{R}.$$

As to the second one, a different insight is needed. It closely resembles $\int \frac{1}{x^2+1}\,\mathrm{d}x$, which leads to the arctan function. To make the connection to the latter integral more explicit, the idea is to factor the 4 out of the denominator:

$$\frac{1}{x^2+4} = \frac{1}{4}\,\frac{1}{\frac{x^2}{4}+1} = \frac{1}{4}\,\frac{1}{\left(\frac{x}{2}\right)^2+1}\ .$$

Now, it only remains to apply the substitution $u = \frac{x}{2}$, $\mathrm{d}u = \frac{1}{2}\mathrm{d}x$:

$$\begin{aligned}\int \frac{1}{x^2+4}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,\frac{1}{2}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{u^2+1}\,\mathrm{d}u = \frac{1}{2}\arctan u + C\\ &= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C\ , \quad C \in \mathbb{R}.\end{aligned}$$

It shall be frequently the case that the integrand cannot be easily expressed in the form $f(\varphi(x))\varphi'(x)$, as in all our previous examples. Often, we have to deal with integrals of the form

$$\int f(\varphi(x)) \cdot g(x)\,\mathrm{d}x\ ,$$

where g is another function of the variable x and no derivative $\varphi'(x)$ appears. Integrals appearing in such a form turn out to be also amenable to the substitution $u = \varphi(x)$, provided that every term appearing in the integrand is replaced by the corresponding term expressed in terms of the new variable u. For this, *we need to assume that we can express x in terms of u, in other words, that φ is invertible*[3]. In this case, we can write $x = \varphi^{-1}(u)$, and thus $\mathrm{d}x = (\varphi^{-1})'(u)\,\mathrm{d}u$, leading to the formal equality[4]

$$\int f(\varphi(x)) \cdot g(x)\,\mathrm{d}x = \int f(u) \cdot g(\varphi^{-1}(u)) \cdot (\varphi^{-1})'(u)\,\mathrm{d}u\ . \tag{1.1.5}$$

Such replacements are justified by Theorem 1.1.1: indeed we may start with the right-hand side of the last-displayed equality and write

$$\begin{aligned}\int f(u) \cdot g(\varphi^{-1}(u)) \cdot (\varphi^{-1})'(u)\,\mathrm{d}u &= \int (f\circ\varphi)(\varphi^{-1}(u)) \cdot g(\varphi^{-1}(u)) \cdot (\varphi^{-1})'(u)\,\mathrm{d}u\\ &= \int (f\circ\varphi\cdot g)(\varphi^{-1}(u))(\varphi^{-1})'(u)\,\mathrm{d}u\ ,\end{aligned}$$

using, in the first equality, the fact that $\varphi(\varphi^{-1}(u)) = u$ by definition of the inverse function. Applying now (1.1.2), we deduce that, since $x = \varphi^{-1}(u)$,

$$\int (f\circ\varphi\cdot g)(\varphi^{-1}(u))(\varphi^{-1})'(u)\,\mathrm{d}u = \left(\int f\circ\varphi(x)\cdot g(x)\,\mathrm{d}x\right)\circ\varphi^{-1}\ .$$

Precomposing both sides with the function $\varphi(x)$, we obtain

$$\left(\int (f\circ\varphi\cdot g)(\varphi^{-1}(u))(\varphi^{-1})'(u)\,\mathrm{d}u\right)\circ\varphi(x) = \int f\circ\varphi(x)\cdot g(x)\,\mathrm{d}x\ ,$$

[3]In case of a definite integral, φ needs to be invertible in the domain of integration under consideration. For indefinite integrals, it is tacitly assumed that the substitution is valid in an appropriate interval where φ is invertible.

[4]As it stands, equality (1.1.5) is meaningless, as the left-hand side is a function of x, whereas the right-hand side is a function of u. However, it is common (and acceptable) practice, whenever applying the substitution method, to write such equalities along the way, provided that in the last step we recall to replace the variable u, which was set to be equal to $\varphi(x)$, by the original function $\varphi(x)$.

which is the formally correct version of the desired equality (1.1.5).

EXAMPLE 1.1.9. Let's go back to Example 1.1.6, and perform a different change of variable: $u = \sqrt{x+3}$. In order to write $\mathrm{d}x$ in terms of the new variable u, we first need to find x in terms of u. Solving $u = \sqrt{x+3}$ for x, we get $u^2 = x+3$ and thus $x = u^2 - 3$, so that $\mathrm{d}x = 2u\mathrm{d}u$. Performing the substitution inside the integral, we obtain

$$\int x^2\sqrt{x+3}\,\mathrm{d}x = \int (u^2-3)^2 \cdot u \cdot 2u\,\mathrm{d}u = \int 2u^2(u^2-3)^2\,\mathrm{d}u\,.$$

The latter we know how to compute, as it is the integral of a polynomial:

$$\int 2u^2(u^2-3)^2\,\mathrm{d}u = \int 2u^6 - 12u^4 + 18u^2\,\mathrm{d}u = \frac{2}{7}u^7 - \frac{12}{5}u^5 + 6u^3 + C\,, \quad C \in \mathbb{R};$$

recalling that $u = \sqrt{x+3}$, we finally get

$$\int x^2\sqrt{x+3}\,\mathrm{d}x = \frac{2}{7}(x+3)^{7/2} - \frac{12}{5}(x+3)^{5/2} + 6(x+3)^{3/2} + C\,, \quad C \in \mathbb{R},$$

which obviously amounts to what we obtained in Example 1.1.6.

In general, there is no unique choice of a good substitution to simplify an integral. Only practice will enable to pick out, each time, the most convenient one.

REMARK 1.1.10. If, instead of the function xe^{x^2}, we were to consider the function e^{x^2} in Example 1.1.4, then performing the same substitution $u = x^2$ is inconclusive. Indeed, we need to express $\mathrm{d}x$ in terms of the new variable u, and to this end we write $x = \sqrt{u}$, leading to $\mathrm{d}x = \frac{1}{2\sqrt{u}}\,\mathrm{d}u$. We have thus transformed

$$\int_1^2 e^{x^2}\,\mathrm{d}x = \int_1^4 \frac{e^u}{2\sqrt{u}}\,\mathrm{d}u\,,$$

which is hardly a friendlier-looking expression. No other substitution possibilities appear, intuitively, to behave better in this example.

As a matter of a fact, it is a mathematical theorem that no primitive[5] of e^{x^2} (over any sub-interval of $\mathbb{R}$) can be expressed using standard composition laws (such as sum, product, powers and so forth) on elementary functions (such as polynomials, exponentials, logarithms, trigonometric functions).

1.2. Integration by parts

In the previous section, we have learned how to bring integrals into a more manageable form using the substitution method, which represents the integral counterpart of the chain rule for derivatives. Another fundamental rule governing the behaviour of derivatives with respect to basic functional operations is the product rule (see Proposition B.2.1 in the Appendix). Once again, the Fundamental Theorem of Calculus indicates that there should be an integral formula, corresponding to the product rule. As we shall presently see, such formula exists and is known as the method of *integration by parts*.

We begin with the version for indefinite integrals

THEOREM 1.2.1 (Integration by parts, indefinite integrals). *Let $u, v\colon (a,b) \to \mathbb{R}$ be two differentiable functions. Then*

$$\int u'(x)v(x)\,\mathrm{d}x = u(x)v(x) - \int u(x)v'(x)\,\mathrm{d}x\,. \tag{1.2.1}$$

Let us be clear on the interpretation of the formula above. The assertion is that the set of anti-derivatives of the product function $u'v$ coincides with the set

$$\{uv - F : F \text{ is an anti-derivative of } uv'\}\,.$$

[5]Observe that such primitives exist since e^{x^2} is a continuous function on the real line.

Proof. Bringing the summand $\int uv' \,\mathrm{d}x$ to the left-hand side of (1.2.1) It is equivalent to show that

$$\int u'(x)v(x) + u(x)v'(x) \,\mathrm{d}x = u(x)v(x) + C\ , \quad C \in \mathbb{R}.$$

By Proposition B.2.1 in the Appendix, $u(x)v(x)$ is an anti-derivative of $u'(x)v(x) + u(x)v'(x)$, whence the conclusion follows from Proposition C.2.2 in the Appendix. $\square$

It is now an easy matter to deduce the version for definite integrals via the Fundamental Theorem of Calculus.

Theorem 1.2.2 (Integration by parts, definite integrals). *Let $c < a \leq b < d$ be real numbers, $u, v\colon (c, d) \to \mathbb{R}$ two continuously differentiable functions. Then*

$$\int_a^b u'(x)v(x) \,\mathrm{d}x = u(x)v(x)\big|_a^b - \int_a^b u(x)v'(x)\ dx\ .$$

Proof. By linearity of the integral and the Fundamental Theorem of Calculus (cf. Theorem C.3.1 in the Appendix),

$$\begin{aligned}\int_a^b u'(x)v(x) \,\mathrm{d}x + \int_a^b u(x)v'(x) \,\mathrm{d}x &= \int_a^b (u'(x)v(x) + u(x)v'(x)) \,\mathrm{d}x = \int_a^b (u(x)v(x))' \,\mathrm{d}x \\ &= u(x)v(x)\big|_a^b\ ;\end{aligned}$$

as desired. $\square$

In pratice, the method of integration by parts is applied when trying to evaluate an integral of the form $\int fg \,\mathrm{d}x$, for some functions f, g, in case we realize that integrating the product Fg', where F is an anti-derivative of f, or the product $f'G$, where G is an anti-derivative of g, is much easier. Let us consider examples of application of the method.

Example 1.2.3. Suppose we want to evaluate

$$\int xe^x \,\mathrm{d}x\ .$$

We realize that *integrating* the factor e^x is elementary, and *differentiating* the other factor x simplifies the expression. In other words, we apply Theorem 1.2.1 with $u'(x) = e^x$, $v(x) = x$; as e^x is an anti-derivative of e^x, we may assume $u(x) = e^x$ and thus get

$$\int xe^x \,\mathrm{d}x = u(x)v(x) - \int u(x)v'(x) \,\mathrm{d}x = xe^x - \int e^x \,\mathrm{d}x = xe^x - e^x + C\ , \quad C \in \mathbb{R}.$$

Example 1.2.4. Let us now compute

$$\int x^2e^x \,\mathrm{d}x\ .$$

The idea is analogous to the previous example: integrating e^x is elementary, and differentiating x^2 lowers the power of x appearing in the integral, thereby simplifying it. Specifically, we apply Theorem 1.2.1 with $u'(x) = e^x$ (so that we may pick $u(x) = e^x$) and $v(x) = x^2$ (so that $v'(x) = 2x$):

$$\int x^2e^x \,\mathrm{d}x = x^2e^x - 2\int xe^x \,\mathrm{d}x\ .$$

For the last displayed integral, we again apply integration by parts exactly as in the previous example. All in all, we get

$$\int x^2e^x \,\mathrm{d}x = x^2e^x - 2(xe^x - e^x) + C\ , \quad C \in \mathbb{R}.$$

EXAMPLE 1.2.5. We would like to evaluate

$$\int x^2 \log x \, \mathrm{d}x \, .$$

Here the observation is that the derivative of $\log x$ is $1/x$, which simplifies with any positive power of x, thereby reducing the integrand to a polynomial. More precisely, we apply Theorem 1.2.1 with $u'(x) = x^2$ (so that we may pick $u(x) = x^3/3$) and $v(x) = \log x$ (with $v'(x) = 1/x$). We obtain

$$\int x^2 \log x \, \mathrm{d}x = \frac{1}{3}x^3 \log x - \frac{1}{3}\int x^3 \frac{1}{x} \, \mathrm{d}x = \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C \, , \quad C \in \mathbb{R}.$$

Before moving on with other examples, here is a symbolic way of memorizing the integration-by-parts formula $\int u'v \, \mathrm{d}x = uv - \int uv' \, \mathrm{d}x$ established in Theorem 1.2.1. Adopting the notation of the substitution method, we may write $u'(x) \, \mathrm{d}x = \mathrm{d}u$ and $v'(x)\mathrm{d}x = \mathrm{d}v$, wherewith the formula becomes

$$\int v \, \mathrm{d}u = uv - \int u \, \mathrm{d}v \, .$$

Such expression is useful to visualize that the method consists in identifying one of the two factors of the product appearing as integrand, namely $\mathrm{d}u$, as the derivative of some function u, which is found via integration, and then remembering that the second factor v needs first to be left as it is, so as to obtain the product uv, and the differentiated in the final integral, resulting in the appearance of $\mathrm{d}v$.

EXAMPLE 1.2.6. At times, integration by parts needs to be combined with other methods, for instance the substitution method. Let's compute, for example,

$$\int x^3 e^{x^2} \, \mathrm{d}x \, .$$

Here we realize that $2xe^{x^2}$ is the derivative (the "$\mathrm{d}u$" in the notation above) of the function e^{x^2}; in other words, we use the substitution $y = x^2$, $\mathrm{d}y = 2x\mathrm{d}x$, to compute

$$\int 2xe^{x^2} \, \mathrm{d}x = \int e^y \, \mathrm{d}y = e^y + C = e^{x^2} + C \, .$$

Secondly, we observe that we can treat the remaining factor $\frac{1}{2}x^2$ in the original integral as the "v" in the above notation:

$$\int x^3 e^{x^2} \, \mathrm{d}x = \int \frac{1}{2}x^2 \cdot 2xe^{x^2} \, \mathrm{d}x = \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2} \, \mathrm{d}x \, .$$

The last displayed integral is, once again, computed via substitution: $\int xe^{x^2} \, \mathrm{d}x = \frac{1}{2}e^{x^2} + C$, so that

$$\int x^3 e^{x^2} \, \mathrm{d}x = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C \, , \quad C \in \mathbb{R}.$$

In certain cases, it is useful to regard the constant function 1 as the function to integrate, as in the following two examples.

EXAMPLE 1.2.7. Suppose we want to evaluate

$$\int \log x \, \mathrm{d}x \, .$$

We realize that we can integrate the factor 1 in the product $1 \cdot \log x$, as it then results in a factor x which simplifies with the derivative of $\log x$. To be precise, we have

$$\int \log x \, \mathrm{d}x = \int 1 \cdot \log x \, \mathrm{d}x = x \log x - \int x \, \frac{1}{x} \, \mathrm{d}x = x \log x - x + C \, , \quad C \in \mathbb{R}.$$

EXAMPLE 1.2.8. We evaluate

$$\int \arctan x \, \mathrm{d}x \, .$$

Once more, here the observation is that the derivative of $\arctan x$ is $(1+x^2)^{-1}$, which behaves well for integration when multiplied by x, which is an anti-derivative of 1. We obtain

$$\begin{aligned}\int \arctan x \, \mathrm{d}x = \int 1 \cdot \arctan x \, \mathrm{d}x &= x \arctan x - \int x \, \frac{1}{1+x^2} \, \mathrm{d}x \\ &= x \arctan x - \frac{1}{2} \log\left(1+x^2\right) + C \, , \quad C \in \mathbb{R},\end{aligned}$$

where in the last inequality we have tacitly used the substitution $u = 1 + x^2$, $\mathrm{d}u = 2x \, \mathrm{d}x$ to evaluate $\int \frac{x}{1+x^2} \, \mathrm{d}x$.

1.3. Trigonometric integrals

1.3.1. Integrals of products of sines and cosines. This subection is devoted to the computation of integrals of the form

$$\int \sin^m(x) \cos^n(x) \, \mathrm{d}x \, ,$$

where $m, n \geq 0$ are integers. Before proceeding with the method, we observe that integrals of this form are typical instances where there might be multiple possible choices of substitution which lead to a simplified integral. For instance, observe that the integral

$$\int \sin x \cos x \, \mathrm{d}x$$

can be either treated with the substitution $u = \sin x$, $\mathrm{d}u = \cos x \mathrm{d}x$, leading to

$$\int u \, \mathrm{d}u = \frac{u^2}{2} + C = \frac{\sin^2(x)}{2} + C \, , \quad C \in \mathbb{R},$$

or, alternatively, with the substitution $u = \cos x$, $\mathrm{d}u = -\sin x \, \mathrm{d}x$, which yields

$$\int -u \, \mathrm{d}u = -\frac{u^2}{2} + C = -\frac{\cos^2(x)}{2} + C \, , \quad C \in \mathbb{R}.$$

Upon closer inspection, the two families of functions are the same: indeed, using Pythagora's theorem $\cos^2(x) + \sin^2(x) = 1$, we can write

$$-\frac{\cos^2(x)}{2} + C = -\frac{1}{2}(1 - \sin^2(x)) + C = \frac{\sin^2(x)}{2} + C - \frac{1}{2} \, ,$$

which, as C varies in $\mathbb{R}$, clearly describes the same family as $\sin^2(x)/2 + C$, $C \in \mathbb{R}$.

We now develop a general technique to evaluate $\int \sin^m(x) \cos^n(x) \, \mathrm{d}x$. We distinguish two cases.

First case: at least one between m and n is odd. We may assume that m is odd; the case where m is even and n is odd is completely symmetric. We can write $m = 2h + 1$, where $h \geq 0$ is an integer. The idea is to factor $\sin^m(x)$ into the product $\sin x \cdot \sin^{2h}(x) = \sin x \cdot (\sin^2(x))^h$, and then use the trigonometric identity $\sin^2(x) = 1 - \cos^2(x)$. We thus obtain

$$\int \sin^m(x) \cos^n(x) \, \mathrm{d}x = \int (1 - \cos^2(x))^h \cos^n(x) \sin x \, \mathrm{d}x \, .$$

At this point, observe that the polynomial $(1-T)^h$, in the variable T, can be expanded as

$$(1-T)^h = a_0 + a_1 T + \cdots a_h T^h$$

for some coefficients $a_i \in \mathbb{R}$, $0 \le i \le h$. Hence, we can expand

$$\begin{aligned}\int \sin^m(x)\cos^n(x)\,\mathrm{d}x &= \int (a_0 + a_1\cos^2(x) + \cdots + a_h\cos^{2h}(x))\cos^n(x)\sin x\,\mathrm{d}x \\ = a_0\int\cos^n(x)\sin x\,\mathrm{d}x &+ a_1\int\cos^{2+n}(x)\sin x\,\mathrm{d}x + \cdots + a_h\int\cos^{2h+n}(x)\sin x\,\mathrm{d}x\ .\end{aligned}$$

Finally, each of the previous integrals is easily computed via the substitution $u = \cos x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$.

Second case: both m and n are even. We can thus write $m = 2h$, $n = 2k$ for some integers $h, k \ge 0$. As a first step, we apply the trigonometric formulas

$$\cos^2(x) = \frac{1+\cos(2x)}{2}\ ,\sin^2(x) = \frac{1-\cos(2x)}{2}\ ,$$

so as to obtain

$$\begin{aligned}\int \sin^m(x)\cos^n(x)\,\mathrm{d}x &= \int \Big(\frac{1-\cos(2x)}{2}\Big)^h\Big(\frac{1+\cos(2x)}{2}\Big)^k\,\mathrm{d}x \\ &= 4^{-(h+k)}\int (1-\cos(2x))^h(1+\cos(2x))^k\,\mathrm{d}x\ .\end{aligned}$$

Now we can expand the polynomial $(1-T)^h(1+T)^k$ as

$$(1-T)^h(1+T)^k = a_0 + a_1T + \cdots + a_{h+k}T^{h+k} \tag{1.3.1}$$

for some coefficients $a_0, \ldots, a_{h+k} \in \mathbb{R}$; hence we get

$$\int \sin^m(x)\cos^n(x)\,\mathrm{d}x = 4^{-(h+k)}\Big(a_0\int 1\,\mathrm{d}x + a_1\int\cos(2x)\,\mathrm{d}x + \cdots + a_{h+k}\int\cos^{h+k}(x)\,\mathrm{d}x\Big)\ .$$

Finally, we integrate each of the summands appearing in the last expression as follows:

- for $\int\cos^p(2x)\,\mathrm{d}x$ with p an odd integer, we apply the method already illustrated above;
- for $\int\cos^p(x)\,\mathrm{d}x$ with p an even integer, we write $p = 2j$ for some integer $j \ge 0$ and apply again the trigonometric formula $\cos^2(2x) = \frac{1+\cos(4x)}{2}$, so as to obtain

  $$\int\cos^p(x)\,\mathrm{d}x = 2^{-j}\int(1+\cos(4x))^j\,\mathrm{d}x\ .$$

  Now we iterate the above procedure, by expanding the polynomial $(1+\cos(4x))^j$. Observe such new polynomial has strictly ower degree with respect to the one in (1.3.1); therefore, after a finite number of steps, only functions of the form $\cos^2(2^\ell x)$, for some integer $\ell \ge 1$, will be left to integrate; it only remains to apply the formula $\cos^2(2^\ell x) = \frac{1+\cos(2^{\ell+1}x)}{2}$ one last time, to reduce matters to an elementary computable integral.

To consolidate the method, we practice it in a list of examples.

Example 1.3.1. We compute

$$\int\sin^3(x)\cos^4(x)\,\mathrm{d}x\ .$$

The sine is raised to an odd power, thus we write

$$\begin{aligned}\int\sin^3(x)\cos^4(x)\,\mathrm{d}x &= \int\sin^2(x)\cos^4(x)\sin x\,\mathrm{d}x = \int(1-\cos^2(x))\cos^4(x)\sin x\,\mathrm{d}x \\ = \int\cos^4(x)\sin x\,\mathrm{d}x &- \int\cos^6(x)\sin x\,\mathrm{d}x = -\frac{\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C\ ,\quad C\in\mathbb{R}.\end{aligned}$$

EXAMPLE 1.3.2. Let's evaluate

$$\int \cos^3(x)\ 2\mathrm{d}x\ .$$

Here the sine function does not appear (rather, for the purposes of the method, it appears with an even power), whereas the cosine function appears with an odd power. We are once more in the first case, and thus we split $\cos^3(x) = \cos x \cos^2(x)$, and apply the fundamental trigonometric identity for $\cos^2(x)$. We have

$$\begin{aligned}\int \cos^3(x)\ \mathrm{d}x &= \int \cos x(1-\sin^2(x))\ \mathrm{d}x = \int \cos x\ \mathrm{d}x - \int \cos x \sin^2(x)\ \mathrm{d}x \\ &= \sin x - \frac{\sin^3(x)}{3} + C\ , \quad C \in \mathbb{R}.\end{aligned}$$

EXAMPLE 1.3.3. We would like to find the solutions, in implicit form, to the separable differential equation

$$y' = \frac{\cos^2(x)}{y}\ .$$

As we have learned, we need to bring every factor depending on y on one side, and every factor depending on x on the other. We do so by multiplying both sides by y, so that we get

$$yy' = \cos^2(x)\ .$$

Taking the integral on both sides, we have

$$\frac{y^2(x)}{2} = \int \cos^2(x)\ \mathrm{d}x\ :$$

It remains to compute the integral on the right-hand side, in which the cosine function is raised to an even power. Hence, we are in the second case, and thus we apply the trigonometric formula for the square of the cosine:

$$\int \cos^2(x)\ \mathrm{d}x = \frac{1}{2}\int 1 + \cos(2x)\ \mathrm{d}x = \frac{x}{2} + \frac{\sin 2x}{4} + C\ , \quad C \in \mathbb{R}.$$

Therefore, we deduce that implicit solutions take the form

$$y^2(x) = x + \frac{\sin(2x)}{2} + C\ , \quad C \in \mathbb{R}.$$

EXAMPLE 1.3.4. Let us now find the solutions, in implicit form, to the separable differential equation

$$y' = y\sin^2(x)\cos^2(x)\ .$$

Diving by y and taking integrals on both sides, we obtain

$$\log|y(x)| = \int \sin^2(x)\cos^2(x)\ \mathrm{d}x\ .$$

As only even powers of sine and cosine appear, we are again in the second case. We thus write

$$\begin{aligned}\int \sin^2(x)\cos^2(x)\ \mathrm{d}x &= \frac{1}{4}\int (1-\cos(2x))(1+\cos(2x))\ \mathrm{d}x = \frac{1}{4}\int (1-\cos^2(2x))\ \mathrm{d}x \\ &= \frac{1}{4}\int \sin^2(2x)\ \mathrm{d}x = \frac{1}{8}\int (1-\cos(4x))\ \mathrm{d}x = \frac{x}{8} - \frac{\sin(4x)}{32} + C\ , \quad C \in \mathbb{R}.\end{aligned}$$

Implicit solutions to the equation take therefore the form

$$\log|y(x)| = \frac{x}{8} - \frac{\sin(4x)}{32} + C\ , \quad C \in \mathbb{R}.$$

Observe that the constant function $y = 0$ is also a solution (ruled out at the beginning as we needed to divide by y).

Trigonometric formulas prove to be useful in evaluating integrals of products of trigonometric functions with different arguments, such as

$$\int \cos(ax)\cos(bx)\ \mathrm{d}x\ , \quad \int \sin(ax)\cos(bx)\ \mathrm{d}x\ , \quad \int \sin(ax)\sin(bx)\ \mathrm{d}x\ ,$$

where $a, b \in \mathbb{R}$. We may assume $b \neq \pm a$, as otherwise, via the properties $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$ we reduce matters to the known integrals $\int \cos^2(x)\ \mathrm{d}x\ ,\ \int \sin^2(x)\ \mathrm{d}x$.

Let us recall the addition formulas for sine and cosine:

$$\begin{aligned} \cos(a+b) &= \cos a \cos b - \sin a \sin b\ , \\ \sin(a+b) &= \sin a \cos b + \cos a \sin b\ . \end{aligned}$$

Observe now that

$$\begin{aligned} \cos(a+b) + \cos(a-b) &= \cos a \cos b - \sin a \sin b + \cos a \cos(-b) - \sin a \sin(-b) \\ &= \cos a \cos b - \sin a \sin b + \cos a \cos b + \sin a \sin b = 2\cos a \cos b\ . \end{aligned}$$

We derive that

$$\cos a \cos b = \frac{\cos(a+b) + \cos(a-b)}{2}\ .$$

Similarly, expanding out $\cos(a-b) - \cos(a+b)$, we derive

$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}\ .$$

Lastly, expanding out $\sin(a+b) + \sin(a-b)$, we get

$$\sin a \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}\ .$$

Using such formulas, we can now evaluate

$$\begin{aligned} \int \cos(ax)\cos(bx)\ \mathrm{d}x &= \frac{1}{2}\int \cos((a+b)x) + \cos((a-b)x)\ \mathrm{d}x \\ &= \frac{1}{2(a+b)}\sin((a+b)x) + \frac{1}{2(a-b)}\sin((a-b)x) + C\ , \quad C \in \mathbb{R}, \end{aligned}$$

$$\begin{aligned} \int \sin(ax)\cos(bx)\ \mathrm{d}x &= \frac{1}{2}\int \sin((a+b)x) + \sin((a-b)x)\ \mathrm{d}x \\ &= -\frac{1}{2(a+b)}\cos((a+b)x) - \frac{1}{2(a-b)}\cos((a-b)x) + C\ , \quad C \in \mathbb{R}, \end{aligned}$$

$$\begin{aligned} \int \sin(ax)\sin(bx)\ \mathrm{d}x &= \frac{1}{2}\int \cos((a-b)x) - \cos((a+b)x)\ \mathrm{d}x \\ &= \frac{1}{2(a-b)}\sin((a-b)x) - \frac{1}{2(a+b)}\sin((a+b)x) + C\ , \quad C \in \mathbb{R}. \end{aligned}$$

1.3.2. Trigonometric substitution. Some integrals do not appear to involve trigonometric functions, but can be evaluated by performing an appropriate substitution with a trigonometric function.

EXAMPLE 1.3.5. Suppose we would like to compute

$$\int \sqrt{4 - x^2}\ \mathrm{d}x\ . \tag{1.3.2}$$

Observe that "standard" substitutions do not lead anywhere meaningful. For instance, putting $u = 4 - x^2$ results in

$$-\int \frac{\sqrt{u}}{\sqrt{4-u}}\ \mathrm{d}u\ ,$$

and substituting $u = \sqrt{4-x^2}$ leads to[6]

$$\int \frac{-2u^2}{\sqrt{4-u^2}}\,\mathrm{d}u\,.$$

Both are hardly simpler to evaluate with respect to the original one. Here the observation is that, if instead of x in (1.3.2) we had $(2\cos(x))^2$, then the whole square root would simplify to $2|\sin x|$, because of the trigonometric identity $1-\cos^2(x) = \sin^2(x)$. Inspired by this observation, we perform the substitution $x = 2\cos u$, $\mathrm{d}x = -2\sin u$, and obtain

$$\int \sqrt{4-x^2}\,\mathrm{d}x = \int \sqrt{4-4\cos^2(u)}(-2\sin u)\,\mathrm{d}u = -4\int \sin u|\sin u|\,\mathrm{d}u\,. \tag{1.3.3}$$

Now notice that, in the end, we will obtain an expression in the variable u for the integral, which we then need to express back as a function of the original variable x. In order to solve $x = 2\cos u$ for u, we need to place ourselves over an open interval where the cosine function is invertible; a natural choice is the interval $(0,\pi)$. Notice that, over this interval, the sine function is positive, and thus we may harmlessly remove the absolute value in (1.3.3):

$$\begin{aligned}\int \sqrt{4-x^2}\,\mathrm{d}x &= -4\int \sin^2(u)\,\mathrm{d}u = -2\int (1-\cos(2u))\,\mathrm{d}u = -2u+\sin(2u)+C\\ &= -2\arccos(x/2)+\sin(2\arccos(x/2))+C\\ &= -2\arccos(x/2)+2\sin(\arccos(x/2))\cos(\arccos(x/2))+C\\ &= -2\arccos(x/2)+x\sqrt{1-x^2/4}+C\\ &= -2\arccos\left(\frac{x}{2}\right)+\frac{x}{2}\sqrt{4-x^2}+C\,,\quad C\in\mathbb{R}\,,\end{aligned}$$

where in the second-to-last equality we have used

$$\cos(\arccos(x/2)) = x/2\,,\quad \sin(\arccos(x/2)) = \sqrt{1-\cos^2(\arccos(x/2))} = \sqrt{1-(x/2)^2}\,,$$

recalling that the sine is positive in the domain where we are considering, so that we have to take the positive square root.

More generally, we can pursue the following strategies.

Integral of $\sqrt{a^2-x^2}$, $a>0$. We substitute $x = a\cos u$, $\mathrm{d}x = -a\sin u\,\mathrm{d}u$. With the same caveat about positivity of the sine function in the chosen domain of invertibility as in the example above, we get

$$\begin{aligned}\int \sqrt{a^2-x^2}\,\mathrm{d}x &= \int \sqrt{a^2-a^2\cos^2(u)}(-a\sin u)\,\mathrm{d}u = -a^2\int \sqrt{1-\cos^2(u)}\sin u\,\mathrm{d}u\\ &= -a^2\int \sin(u)|\sin(u)|\,\mathrm{d}u = -a^2\int \sin^2(u)\,\mathrm{d}u = -\frac{a^2}{2}\int (1-\cos(2u))\,\mathrm{d}u\\ &= -\frac{a^2}{2}\left(u-\frac{\sin(2u)}{2}\right)+C = -\frac{1}{2}\,a^2\arccos\left(\frac{x}{a}\right)+\frac{a^2}{4}\sin\left(2\arccos\left(\frac{x}{a}\right)\right)+C\\ &= -\frac{a^2}{2}\left(u-\frac{\sin(2u)}{2}\right)+C = -\frac{1}{2}\,a^2\arccos\left(\frac{x}{a}\right)+\frac{a^2}{2}\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}+C\\ &= -\frac{a^2}{2}\left(u-\frac{\sin(2u)}{2}\right)+C = -\frac{1}{2}\,a^2\arccos\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+C\,,\quad C\in\mathbb{R}.\end{aligned}$$

REMARK 1.3.6. Observe that it is equally possible to apply the substitution $x = a\sin u$, $\mathrm{d}x = a\cos u\,\mathrm{d}u$. The same steps carried out above, with the appropriate modifications, allow to compute the given integral.

[6]In both cases we are implicitly choosing the positive square root when expressing x as a function of u, but this choice is completely arbitrary and thus constitutes an additional nuisance in performing such substitutions in this case.

Integral of $\sqrt{a^2+x^2}$, $a>0$. In this case, it is convenient to apply the substitution $x = a\tan u$, $\mathrm{d}x = a\sec^2(u)\,\mathrm{d}u$. In this way, we obtain

$$\int \sqrt{a^2+x^2}\,\mathrm{d}x = \int \sqrt{a^2(1+\tan^2(u))}\,a\sec^2(u)\,\mathrm{d}u = a^2\int \sec^2(u)|\sec u|\,\mathrm{d}u\,.$$

The tangent function is well defined and invertible on the interval $(-\pi/2,\pi/2)$; on such interval, the cosine function is positive, thus we may remove the absolute value and get

$$\int \sqrt{a^2+x^2}\,\mathrm{d}x = a^2\int \sec^3(u)\,\mathrm{d}u = a^2\int \frac{1}{\cos^3(u)}\,\mathrm{d}u\,.$$

We have learned how to integrate positive integral powers of cosine and sine. What about negative powers? In this specific case, we notice that multiplying numerator and denominator by $\cos u$ leads to

$$\int \sqrt{a^2+x^2}\,\mathrm{d}x = a^2\int \frac{\cos(u)}{(1-\sin^2(u))^2}\,\mathrm{d}u = a^2\int \frac{1}{(1-v^2)^2}\,\mathrm{d}v\,,$$

where in the last step we applied a further substitution $v=\sin u$.

It turns out that there is a standard procedure to evaluate integrals of rational functions such as the one appearing in the last displayed integral. This is the subject of Section 1.4.

Integral of $\sqrt{x^2-a^2}$, $a>0$. Here the convenient substitution to perform is $x = a\sec u = \frac{a}{\cos u}$, $\mathrm{d}x = a\frac{\sin u}{\cos^2 u}\,\mathrm{d}u$, which leads to

$$\begin{aligned}\int \sqrt{x^2-a^2}\,\mathrm{d}x &= \int \sqrt{\frac{a^2}{\cos^2(u)}-a^2}\;a\,\frac{\sin u}{\cos^2(u)}\,\mathrm{d}u = a^2\int \sqrt{\frac{1-\cos^2(u)}{\cos^2(u)}}\,\frac{\sin u}{\cos^2(u)}\,\mathrm{d}u\\ &= a^2\int \left|\frac{\sin u}{\cos u}\right|\frac{\sin u}{\cos^2(u)}\,\mathrm{d}u\,.\end{aligned}$$

Observe now that, as in the previously examined cases, we need to choose an open interval over which we can unambigously express u in terms of x, namely over which the sec function is invertible. Observe that this is the case over any open interval $(k\pi/2, \pi/2+k\pi/2)$, as k ranges over the integers. We may choose, for instance, the interval $(0,\pi/2)$, where both the sine and the cosine function are positive. With this choice we get therefore

$$\begin{aligned}\int \sqrt{x^2-a^2}\,\mathrm{d}x &= a^2\int \frac{\sin^2(u)}{\cos^3(u)}\,\mathrm{d}u = a^2\int \frac{1-\cos^2(u)}{\cos^3(u)}\,\mathrm{d}u = a^2\Big(\int \frac{1}{\cos^3(u)}\,\mathrm{d}u - \int\frac{1}{\cos u}\,\mathrm{d}u\Big)\\ &= a^2\Big(\int \frac{1}{(1-v^2)^2}\,\mathrm{d}v - \int \frac{\cos u}{1-\sin^2(u)}\,\mathrm{d}u\Big)\\ &= a^2\Big(\int \frac{1}{(1-v^2)^2}\,\mathrm{d}v - \int \frac{1}{1-v^2}\,\mathrm{d}v\Big)\,,\end{aligned}$$

where in the second-to-last step we have used the substitution $v=\sin(u)$ for the first integral, as in the case of $\sqrt{a^2+x^2}$ examined above, and multiplied numerator and denominator by $\cos u$ in the second integral, whereas in the last step we again used the substitution $v=\sin u$.

Once again, we shall explore how to evaluate integrals such as those appearing in the last displayed expression in Section 1.4.

Further examples.

EXAMPLE 1.3.7. Let's compute

$$\int \frac{1}{(4-x^2)^{3/2}}\,\mathrm{d}x\,.$$

We use the substitution $x=2\sin u$, $\mathrm{d}x = 2\cos u\,\mathrm{d}u$, which results in

$$\int \frac{1}{(4-x^2)^{3/2}}\,\mathrm{d}x = 2\int \frac{\cos u}{(4-4\sin^2(u))\sqrt{4-4\sin^2(u)}}\,\mathrm{d}u = \frac{1}{4}\int\frac{\cos u}{\cos^2(u)|\cos(u)|}\,\mathrm{d}u\,.$$

We choose to work over the interval $(-\pi/2, \pi/2)$, over which the sine function is invertible and thus we may at the end express u as a function of x. Over such interval, the cosine function is positive, and hence we may harmlessly remove the absolute value. We get

$$\int \frac{1}{(4-x^2)^{3/2}}\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{\cos^2(u)}\,\mathrm{d}u = \frac{1}{4}\tan u + C = \frac{1}{4}\frac{\sin(\arcsin(x/2))}{\cos(\arcsin(x/2))} + C$$
$$= \frac{1}{16}\frac{x}{\sqrt{4-x^2}} + C\ , \quad C \in \mathbb{R}.$$

Example 1.3.8. We now turn to the integral

$$\int \frac{x}{(4-x^2)^{3/2}}\,\mathrm{d}x\ .$$

This is taken care of by the same substitution as before, $x = 2\sin u$, $\mathrm{d}x = 2\cos u\ \mathrm{d}u$. The integral becomes

$$\int \frac{x}{(4-x^2)^{3/2}}\,\mathrm{d}x = \frac{1}{2}\int \frac{\sin u}{\cos^2(u)}\,\mathrm{d}u\ ,$$

which can be treated with the further substitution $v = \cos u$, leading to

$$\int \frac{x}{(4-x^2)^{3/2}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{1}{v^2}\,\mathrm{d}v = \frac{1}{2v} + C = \frac{1}{2\cos u} + C = \frac{1}{2\cos(\arcsin(x/2))} + C$$
$$= \frac{1}{\sqrt{4-x^2}} + C\ , \quad C \in \mathbb{R}.$$

Observe that there is even a faster way to reach the answer, by simply using the substitution $4 - x^2 = u$: this directly leads to

$$\int \frac{x}{(4-x^2)^{3/2}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{1}{u^{3/2}}\,\mathrm{d}u = \frac{1}{\sqrt{u}} + C = \frac{1}{\sqrt{4-x^2}} + C\ , \quad C \in \mathbb{R}.$$

Example 1.3.9. We want to find implicit solutions to the separable equation

$$y'(x) = \frac{y(x)}{(9+x^2)^{5/2}}\ .$$

Dividing by $y(x)$ and integrating, we get

$$\log|y(x)| = \int \frac{1}{(9+x^2)^{5/2}}\,\mathrm{d}x\ ,$$

hence it remains to evaluate the last displayed integral. For that, we use the substitution $x = 3\tan u$, $\mathrm{d}x = \frac{3}{\cos^2(u)}\,\mathrm{d}u$, which transforms the original integral into

$$3\int \frac{1}{\cos^2(u)(9+9\tan^2(u))^{5/2}}\,\mathrm{d}u = \frac{1}{81}\int \frac{1}{\cos^2(u)(\frac{1}{\cos^5(u)})}\,\mathrm{d}u = \frac{1}{81}\int \cos^3(u)\,\mathrm{d}u =$$
$$\frac{1}{81}\int \cos u(1-\sin^2(u))\,\mathrm{d}u\ .$$

We conclude by substituting $v = \sin u$, $\mathrm{d}v = \cos u\ \mathrm{d}u$, so as to get

$$\int \frac{1}{(9+x^2)^{5/2}}\,\mathrm{d}x = \frac{1}{81}\int 1 - v^2\,\mathrm{d}v = \frac{v}{81} - \frac{v^3}{243} + C = \frac{\sin u}{81} - \frac{\sin^3(u)}{243} + C$$
$$= \frac{1}{81}\sin(\arctan(x/3)) - \frac{1}{243}\sin^3(\arctan(x/3)) + C, \quad C \in \mathbb{R}.$$

In order to express the sine function in terms of the tangent function, we observe that $\tan^2(y) + 1 = \frac{1}{\cos^2(y)} = \frac{1}{1-\sin^2(y)}$, from which we obtain $\sin y = \frac{\tan(y)}{\sqrt{1+\tan^2(y)}}$ (we choose the positive square root in the numerator as the sine and the tangent function have the same sign over the reference interval $-\pi/2, \pi/2)$. Finally, we get implicit solutions of the form

$$\log|y(x)| = \frac{1}{81}\frac{x/3}{\sqrt{1+x^2/9}} - \frac{1}{243}\frac{x^3/27}{(1+x^2/9)^{3/2}} + C\ , \quad C \in \mathbb{R}.$$

Observe that the function $y(x) = 0$ is also a solution to the original equation.

EXAMPLE 1.3.10. Let's find implicit solutions to

$$y'(x) = \frac{xy^2(x)}{(9+x^2)^{5/2}}\ .$$

Observing that $y(x) = 0$ is a solution, we divide both sides by $y^2(x)$ and integrate, thus obtaining

$$-\frac{1}{y(x)} = \int \frac{x}{(9+x^2)^{5/2}}\ \mathrm{d}x\ .$$

For the integral on the right-hand side, we can use the substitution $u = 9 + x^2$, which directly allows to compute

$$\int \frac{x}{(9+x^2)^{5/2}}\ \mathrm{d}x = \frac{1}{2}\int \frac{1}{u^{5/2}}\ \mathrm{d}u = -\frac{1}{3}u^{-3/2} + C = -\frac{1}{3(9+x^2)^{3/2}} + C\ , \quad C \in \mathbb{R}.$$

To practice trigonometric substitution, we also proceed with the alternative substitution $x = 3\tan u$, $\mathrm{d}x = \frac{3}{\cos^2(u)}\ \mathrm{d}u$, which gives

$$\begin{aligned}\int \frac{x}{(9+x^2)^{5/2}}\ \mathrm{d}x &= \frac{1}{27}\int \tan u \cos^3(u)\ \mathrm{d}u = \frac{1}{27}\int \sin u \cos^2(u)\ \mathrm{d}u = -\frac{1}{81}\cos^3(u) + C \\ &= -\frac{1}{81}\cos^3(\arctan(x/3)) + C = -\frac{1}{3(9+x^2)^{3/2}} + C\ , \quad C \in \mathbb{R},\end{aligned}$$

using that $\tan^2(y) + 1 = \frac{1}{\cos^2(y)}$, so that $\cos(y) = \frac{1}{\sqrt{\tan^2(y)+1}}$. Implicit solutions are thus of the form

$$-\frac{1}{y(x)} = -\frac{1}{3(9+x^2)^{3/2}} + C\ , \quad C \in \mathbb{R}.$$

1.4. Partial fraction decomposition

This section is devoted to development of a general method to integrate *rational functions*. A **rational function** is a quotient of two polynomial functions, that is, it's a function f of the form

$$F(x) = \frac{a_r x^r + \cdots + a_1 x + x_0}{b_s x^s + \cdots + b_1 x + b_0}\ , \tag{1.4.1}$$

where $r, s \geq 0$ are integers, $a_0, \ldots, a_r, b_0, \ldots, b_s$ are given real numbers, and the denominator is not the zero polynomial.

We have already seen how to integrate some rational functions: for instance, we know that

$$\int \frac{1}{x-a}\ \mathrm{d}x = \log|x-a| + C\ , \quad C \in \mathbb{R}$$

for every $a \in \mathbb{R}$, that

$$\int \frac{1}{x^2+a^2}\ \mathrm{d}x = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\ \mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C\ , \quad C \in \mathbb{R},$$

for every $a \in \mathbb{R}$, $a \neq 0$, and that

$$\int \frac{1}{x^m}\ \mathrm{d}x = \frac{1}{1-m}x^{1-m} + C\ , \quad C \in \mathbb{R},$$

for every integer $m \geq 2$.

The method of *partial fraction decomposition* allows to evaluate integrals of any rational function, by ultimately reducing matters to one of the already known integrals mentioned above. This is achieved by decomposing the original rational function as a sum of rational functions of lower degree, and iterating the procedure until we reach a quotient of a polynomial of degree at most one divided by a polynomial of degree at most two. Let's illustrate the various steps of the method carefully.

First step: reducing to a rational function with denominator of higher degree. When computing the integral $\int F(x)\,\mathrm{d}x$ for a function f of the form given in (1.4.1), it is always possible to reduce matters to the case where the polynomial in the numerator has strictly lower degree with respect to the one appearing in the denominator; in this case we say that the rational function is **proper**. This is due to the familiar algorithm of division with remained for polynomials.

PROPOSITION 1.4.1 (Division with remainder for polynomials). *Let $f(x), g(x)$ be two polynomial functions, where $g(x)$ has degree $\deg g(x) \geq 1$. Then there exists uniquely determined polynomials $q(x), r(x)$, with $\deg r(x) < \deg g(x)$, such that*

$$f(x) = q(x)g(x) + r(x)\ .$$

Suppose thus $F(x) = f(x)/g(x)$ for two polynomial functions $f(x), g(x)$. We may clearly assume that $\deg g(x) \geq 1$, as otherwise $F(x)$ is a polynomial itself, which we already know how to integrate. We are thus in the setting of Proposition 1.4.1; if $\deg f(x) \geq \deg g(x)$, then we apply the proposition, or in more elementary terms we divide the polynomial f by the polynomial g, finding thus polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)g(x) + r(x)$ and $\deg r(x) < \deg g(x)$; $r(x)$ is called the remainder of the division of f by g. We can thus write

$$F(x) = \frac{f(x)}{g(x)} = \frac{q(x)g(x)}{g(x)} + \frac{r(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}\ ;$$

now we know how to integrate the polynomial function $q(x)$, and we are thus left with the problem of computing $\int \frac{r(x)}{g(x)}\,\mathrm{d}x$, where now the numerator has strictly lower degree that the denominator.

Decomposing into partial fractions. In light of the previous paragraph, we can assume we are dealing with an integral $\int \frac{f(x)}{g(x)}\,\mathrm{d}x$, where $\deg f(x) < \deg g(x)$. We would now like to decompose the fraction $\frac{f(x)}{g(x)}$ as a sum of fractions of the form $\frac{a(x)}{b(x)}$, where $\deg a(x) \leq 1$ and $\deg b(x) \leq 2$. The fundamental abstract result enabling us to do so is the following.

PROPOSITION 1.4.2 (Decomposition of a polynomial into irreducible factors). *Let $g(x)$ be a polynomial function. Then there are polynomial functions $g_1(x), \ldots, g_r(x)$, with $r \geq 1$ an integer, such that the following properties hold:*

(1) $\deg g_i(x) = 1$ or $\deg g_2(x) = 2$ for every $1 \leq i \leq r$;
(2) $g(x)$ can be factored as a product

$$g(x) = g_1(x)\cdots g_r(x)\ ;$$

(3) all $g_i(x)$ are irreducible, namely either $\deg g_i(x) = 1$, or $\deg g_i(x) = 2$ and $g_i(x)$ cannot be further factored as a product of two polynomials of degree 1.

The proof of this proposition involves complex numbers, which will be introduced in Chapter **??**, and falls in any event outside the purview of this course.

EXAMPLE 1.4.3. The polynomial $x^2 + 1$ is irreducible. Indeed, polynomials of degree two are reducible if and only if they admit a (real) root, which is clearly not the case for $x^2 + 1$. On the other hand, the polynomial $x^2 - 1$ is not irreducible, as it can be factored as $(x-1)(x+1)$.

EXAMPLE 1.4.4. Let's consider the polynomial $g(x) = x^3 - 1$, which has degree 3. We want to find a factorization into irreducible polynomials. For this, we start by noticing that 1 is a root of g, hence $g(x)$ must be divisible by $x - 1$. Performing the division algorithm for polynomials, we find that

$$g(x) = x^3 - 1 = (x - 1)(x^2 + x + 1)\,.$$

The polynomial x^2+x+1 is irreducible, since it has no roots (over the real numbers). Thus, the one indicated above is the desired factorization of $g(x)$ as a product of irreducible polynomials.

We are now in the following situation: we wish to compute the integral of

$$\frac{f(x)}{g(x)} = \frac{f(x)}{g_1(x)\cdots g_r(x)}\ ,$$

where $g_1(x)\cdots g_r(x)$ is a factorization of the denominator $g(x)$ into irreducible polynomials, of degree at most two. Up to reordering the factors, we may assume that $\deg g_1 = \cdots = \deg g_s = 1$, $\deg g_{s+1} = \cdots = \deg g_r = 2$ for some $0 \le s \le r$. It is also notationally convenient, for what comes next, to group identical factors together, and thus write upon renaming everything

$$\frac{f(x)}{g(x)} = \frac{f(x)}{g_1(x)^{k_1}\cdots g_s(x)^{k_s} g_{s+1}(x)^{k_{s+1}}\cdots g_r(x)^{k_r}}$$

for some integers $k_1, \ldots, k_r \ge 1$, with all the polynomials $g_i(x)$ essentially different from each other (meaning no one is a multiple of any other by a constant). We now appeal to the following general fact, whose proof is elementary but tedious, and thus omitted.

PROPOSITION 1.4.5 (General partial fraction decomposition). *Let*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{g_1(x)^{k_1}\cdots g_r(x)^{k_r}}$$

be a rational function as above. Then there are real numbers

$$A_1^{(1)}, \ldots, A_1^{(k_1)}, A_2^{(1)}, \ldots, A_2^{(k_2)}, \ldots, A_s^{(1)}, \ldots, A_s^{(k_s)}$$

and

$$B_{s+1}^{(1)}, C_{s+1}^{(1)}, \ldots, B_{s+1}^{(k_{s+1})}, C_{s+1}^{(k_{s+1})}, \ldots, B_r^{(1)}, C_r^{(1)}, \ldots, B_r^{(k_r)} C_r^{(k_r)}$$

such that

$$\begin{aligned}\frac{f(x)}{g(x)} =& \frac{A_1^{(1)}}{g_1(x)} + \frac{A_1^{(2)}}{g_1(x)^2} + \cdots + \frac{A_1^{(k_1)}}{g_1(x)^{k_1}} + \cdots + \frac{A_s^{(1)}}{g_s(x)} + \cdots + \frac{A_s^{(k_s)}}{g_s(x)^{k_s}}\\ &+ \frac{B_{s+1}^{(1)}x + C_{s+1}^{(1)}}{g_{s+1}(x)} + \cdots + \frac{B_{s+1}^{(k_{s+1})}x + C_{s+1}^{(k_{s+1})}}{g_{s+1}(x)^{k_{s+1}}} + \cdots + \frac{B_r^{(1)}x + C_r^{(1)}}{g_r(x)} + \cdots + \frac{B_r^{(k_r)}x + C_r^{(k_r)}}{g_r(x)^{k_r}}\ .\end{aligned}$$

Concisely put, the import of the proposition is the following: just as any non-constant polynomial can be factored into the product of polynomials of degree at most two, any proper rational function can be decomposed into the sum of proper rational functions where the denominator is an integral power of an irreducible polynomial, of degree at most two, and the degree of the numerator is at most one.

Evaluating the remaining integrals with the methods learned thus far. We are now reduced with the tast of evaluating integrals of proper rational functions of one of the following forms:

- for every $a \in \mathbb{R}$,
$$\int \frac{1}{x-a}\,\mathrm{d}x = \log|x-a| + C\ , \quad C \in \mathbb{R}\ ;$$
- for every integer $j \ge 2$ and $a \in \mathbb{R}$,
$$\int \frac{1}{(x-a)^j}\,\mathrm{d}x = \frac{1}{1-j}(x-a)^{1-j} + C\ , \quad C \in \mathbb{R};$$

- for every $a \in \mathbb{R}$,
$$\int \frac{x}{x^2+a^2}\,\mathrm{d}x = \frac{1}{2}\log\left(x^2+a^2\right) + C\,, \quad C \in \mathbb{R};$$
- for every integer $j \geq 2$ and every $a \in \mathbb{R}$,
$$\int \frac{x}{(x^2+a^2)^j} = \frac{1}{2(1-j)}(x^2+a^2)^{1-j} + C\,, \quad C \in \mathbb{R};$$
- for every $a \in \mathbb{R}$,
$$\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C\,, \quad C \in \mathbb{R};$$
- for every integer $j \geq 2$ and every $a \in \mathbb{R}$,
$$\begin{aligned}\int \frac{1}{(x^2+a^2)^j}\,\mathrm{d}x &= a\int \frac{1}{\cos^2(u)(a^2\tan^2(u)+a^2)^j}\,\mathrm{d}u = a^{1-2j}\int \frac{1}{\cos^2(u)\left(\frac{1}{\cos^2(u)}\right)^j}\,\mathrm{d}u \\ &= a^{1-2j}\int \cos^{2(j-1)}(u)\,\mathrm{d}u\,,\end{aligned}$$
which we know how to integrate from Section 1.3.1.

Observe that we have indeed exhausted all possibilities. In particular, if $\frac{Ax+B}{ax^2+bx+c}$ is a proper rational function where the denominator is an irreducible polynomial of degree two, then we can *complete the square* to reduce matters to one of the integrals mentioned above. To be precise, we write (assuming without loss of generality that $a > 0$, otherwise put a minus sign in front of the rational function and change sign to all coefficients of the polynomial)

$$ax^2+bx+c = \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + c - \frac{b^2}{4a}\,,$$

where the irreducibility assumption ensures that $c - b^2/4a > 0$. Thus we may write $c - b^2/4a = \alpha^2$ for some $\alpha > 0$; then

$$\begin{aligned}\int \frac{Ax+B}{ax^2+bx+c}\,\mathrm{d}x &= \int \frac{Ax+B}{\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2+\alpha^2}\,\mathrm{d}x = \frac{1}{a}\int \frac{A(u-\frac{b}{2\sqrt{a}})+\sqrt{a}B}{u^2+\alpha^2}\,\mathrm{d}u \\ &= \frac{A}{a}\int \frac{u}{u^2+\alpha^2}\,\mathrm{d}u + \frac{\sqrt{a}B - \frac{b}{2\sqrt{a}}A}{a}\int \frac{1}{u^2+\alpha^2}\,\mathrm{d}u\,,\end{aligned}$$

where we have applied the substitution $u = \sqrt{a}x + \frac{b}{2\sqrt{a}}$, $\mathrm{d}u = \sqrt{a}\mathrm{d}x$. The integrals appearing in the last displayed term have already been examined above.

A similar procedure deals with the case $\frac{Ax+B}{(ax^2+bx+c)^j}$, where now $j \geq 1$ is an arbitrary integer and ax^2+bx+c is an irreducible polynomial.

1.4.1. Examples of partial fraction decomposition. Having illustrated the general method, we now see it in action in concrete examples.

Example 1.4.6. We compute
$$\int \frac{1}{x^2-1}\,\mathrm{d}x\,.$$
The integrand is a proper function, hence the first step is to factor the denominator, which yields $x^2-1 = (x-1)(x+1)$. Hence the partial fraction decomposition takes necessarily the form
$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

for some real numbers A and B to be determined. In order to find those, we sum the two fractions on the right-hand side of the last displayed equality, there by obtaining

$$\frac{1}{x^2-1} = \frac{A(x+1)+B(x-1)}{x^2-1} = \frac{(A+B)x+A-B}{x^2-1} \,.$$

Recall the fundamental fact that *two polynomials* $a_r x^r + \cdots + a_1 x + a_0$, $b_s x^s + \cdots + b_1 x + b_0$ *are equal if and only if* $r = s$ *and* $a_i = b_i$ *for every* $0 \le i \le r$. Therefore, we must have

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} .$$

From the first we deduce $B = -A$, and plugging this into the second we obtain $2A = 1$, that is $A = 1/2$, and thus $B = -1/2$. We have thus obtained the decomposition

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) .$$

Hence

$$\int \frac{1}{x^2-1}\,\mathrm{d}x = \frac{1}{2}\left(\int \frac{1}{x-1}\,\mathrm{d}x - \int \frac{1}{x+1}\,\mathrm{d}x\right) = \frac{1}{2}(\log|x-1| - \log|x+1|) + C\,, \quad C \in \mathbb{R}.$$

Example 1.4.7. We compute

$$\int \frac{2x+3}{x^2-1}\,\mathrm{d}x \,.$$

We have the same denominator as in the last example, and thus again the decomposition must take the form

$$\frac{2x+3}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

for some real numbers A and B to be determined. We impose

$$\frac{2x+3}{x^2-1} = \frac{A(x+1)+B(x-1)}{x^2-1} = \frac{(A+B)x+A-B}{x^2-1} \,,$$

so that we we must have

$$\begin{cases} A+B=2 \\ A-B=3 \end{cases} .$$

From the first we deduce $B = 2 - A$, and plugging this into the second we obtain $2A = 5$, that is $A = 5/2$, and thus $B = -1/2$. We have thus obtained the decomposition

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{5}{x-1} - \frac{1}{x+1}\right) .$$

Hence

$$\int \frac{2x+3}{x^2-1}\,\mathrm{d}x = \frac{1}{2}\left(\int \frac{5}{x-1}\,\mathrm{d}x - \int \frac{1}{x+1}\,\mathrm{d}x\right) = \frac{1}{2}(5\log|x-1| - \log|x+1|) + C\,, \quad C \in \mathbb{R}.$$

Example 1.4.8. Let's now compute

$$\int \frac{3x+5}{x^2+2x+1}\,\mathrm{d}x \,.$$

Once again, the rational function is proper and thus we can directly proceed to factoring the denominator. This time, it is the square of a linear polynomial: $x^2 + 2x + 1 = (x+1)^2$. As a result, the partial fraction decomposition needs to take the form

$$\frac{3x+5}{x^2+2x+1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{x^2+2x+1} = \frac{Ax+A+B}{x^2+2x+1}$$

for some A and B, which are found by solving the linear system

$$\begin{cases} A=3 \\ A+B=5 \end{cases} ,$$

which delivers $A = 3$, $B = 2$. Therefore,

$$\int \frac{3x+5}{x^2+2x+1} = 3\int \frac{1}{x+1}\,\mathrm{d}x + 2\int \frac{1}{(x+1)^2}\,\mathrm{d}x = 3\log|x+1| - \frac{2}{x+1} + C\ , \quad C \in \mathbb{R}.$$

Example 1.4.9. Suppose now we want to evaluate

$$\int \frac{x^4+7x+11}{x^2(x+1)^3}\,\mathrm{d}x\ .$$

The decomposition of the denominator into irreducible factors is already given, whence the partial fraction decomposition must take the form

$$\frac{x^4+7x+11}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

for some real numbers A, B, C, D, E. We have

$$\begin{aligned}\frac{x^4+7x+11}{x^2(x+1)^3} &= \frac{Ax(x+1)^3 + B(x+1)^3 + Cx^2(x+1)^2 + Dx^2(x+1) + Ex^2}{x^2(x+1)^3}\\ &= \frac{(A+C)x^4 + (3A+B+2C+D)x^3 + (3A+3B+C+D+E)x^2 + (A+3B)x + B}{x^2(x+1)^3}\ ,\end{aligned}$$

so that we need to solve

$$\begin{cases} A + C = 1 \\ 3A + B + 2C + D = 0 \\ 3A + 3B + C + D + E = 0 \\ A + 3B = 7 \\ B = 11 \end{cases} .$$

From the fourth equation we get $A = 7 - 3B = 7 - 33 = -26$, so that from the first one we can deduce $C = 1 - A = 27$, from the second one $D = -3A - B - 2C = 78 - 11 - 54 = 13$, and finally from the third one $E = -3A - 3B - C - D = 78 - 33 - 27 - 13 = 5$. We can now compute

$$\begin{aligned}\int \frac{x^4+7x+11}{x^2(x+1)^3}\,\mathrm{d}x &= -26\int \frac{1}{x}\,\mathrm{d}x + 11\int \frac{1}{x^2}\,\mathrm{d}x + 27\int \frac{1}{x+1}\,\mathrm{d}x + 13\int \frac{1}{(x+1)^2}\,\mathrm{d}x\\ &\quad + 5\int \frac{1}{(x+1)^3}\,\mathrm{d}x\\ &= -26\log|x| - \frac{11}{x} + 27\log|x+1| - \frac{13}{x+1} - \frac{5}{2(x+1)^2} + C\ , \quad C \in \mathbb{R}.\end{aligned}$$

Example 1.4.10. Let's compute

$$\int \frac{6x-3}{x^2+4x+5}\,\mathrm{d}x\ .$$

The denominator has no real roots, as it can be written as $(x+2)^2+1$. In this case, the most convenient way to decompose the original fraction is to make the derivative of the denominator, namely $2(x+2)$, appear in the numerator, and leave the rest as a constant: specifically, we write

$$\frac{6x-3}{x^2+4x+5} = \frac{6(x+2)}{x^2+4x+5} - \frac{15}{x^2+4x+5}\ . \tag{1.4.2}$$

The integral of the first term is taken care of by an elementary substitution:

$$\int \frac{6(x+2)}{x^2+4x+5}\,\mathrm{d}x = 3\int \frac{2(x+2)}{(x+2)^2+1}\,\mathrm{d}x = 3\log(x^2+4x+5) + C\ , \quad C \in \mathbb{R},$$

where notice that there is no need to put the absolute value for the argument of the logarithm as it is always positive. As far as the second summand in (1.4.2) is concerned, we have

$$\int \frac{15}{x^2+4x+5}\,\mathrm{d}x = 15\int \frac{1}{(x+1)^2+1}\,\mathrm{d}x = 15\arctan(x+1)+C\ , \quad C\in\mathbb{R}.$$

All in all, we have

$$\int \frac{6x-3}{x^2+4x+5}\,\mathrm{d}x = 3\log(x^2+4x+5) - 15\arctan(x+1)+C\ , \quad C\in\mathbb{R}.$$

EXAMPLE 1.4.11. We wish to evaluate

$$\int \frac{x^3-1}{(x^2+4x+5)^2}\,\mathrm{d}x\ .$$

The denominator is the square of an irreducible polynomial, whence the partial fraction decomposition must take the form

$$\begin{aligned}\frac{x^3-1}{(x^2+4x+5)^2} &= \frac{Ax+B}{x^2+4x+5} + \frac{Cx+D}{(x^2+4x+5)^2} = \frac{(Ax+B)(x^2+4x+5)+Cx+D}{(x^2+4x+5)^2}\\ &= \frac{Ax^3+(4A+B)x^2+(5A+4B+C)x+5B+D}{(x^2+4x+5)^2}\ .\end{aligned}$$

Therefore, we must have

$$\begin{cases} A=1\\ 4A+B=0\\ 5A+4B+C=0\\ 5B+D=-1\end{cases},$$

which gives $B=-4A=-4$, $C=-5A-4B=-5+16=11$, $D=-1-5B=-1+20=19$. Thus

$$\int \frac{x^3-1}{(x^2+4x+5)^2}\,\mathrm{d}x = \int \frac{x-4}{x^2+4x+5}\,\mathrm{d}x + \int \frac{11x+19}{(x^2+4x+5)^2}\,\mathrm{d}x\ .$$

For the first summand, we argue as in the previous example, and write

$$\begin{aligned}\int \frac{x-4}{x^2+4x+5}\,\mathrm{d}x &= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\,\mathrm{d}x - 6\frac{1}{(x+1)^2+1}\,\mathrm{d}x\\ &= \frac{1}{2}\log(x^2+4x+5) - 6\arctan(x+1)+C\ , \quad C\in\mathbb{R}.\end{aligned}$$

To deal with the remaining term, we argue in a similar manner and write

$$\begin{aligned}\int \frac{11x+19}{(x^2+4x+5)^2}\,\mathrm{d}x &= \frac{11}{2}\int \frac{2x+4}{(x^2+4x+5)^2}\,\mathrm{d}x - 3\int \frac{1}{(x^2+4x+5)^2}\,\mathrm{d}x\\ &= -\frac{11}{2(x^2+4x+5)} - 3\int \frac{1}{(x^2+4x+5)^2}\,\mathrm{d}x\ .\end{aligned}$$

The last integral is taken care of by a trigonometric substitution: setting first $u=x+1$ and then $u=\tan v$, we have

$$\begin{aligned}\int \frac{1}{((x+1)^2+1)^2}\,\mathrm{d}x &= \int \frac{1}{(u^2+1)^2}\,\mathrm{d}u = \int \frac{1}{\cos^2(v)(\tan^2(v)+1)^2}\,\mathrm{d}v\\ &= \int \cos^2(v)\,\mathrm{d}v = \frac{1}{2}\Big(v+\frac{1}{2}\sin 2v\Big)+C\\ &= \frac{1}{2}\arctan u + \frac{1}{4}\sin(2\arctan u)+C\\ &= \frac{1}{2}\arctan(x+1) + \frac{1}{4}\sin(2\arctan(x+1))+C\ , \quad C\in\mathbb{R}.\end{aligned}$$

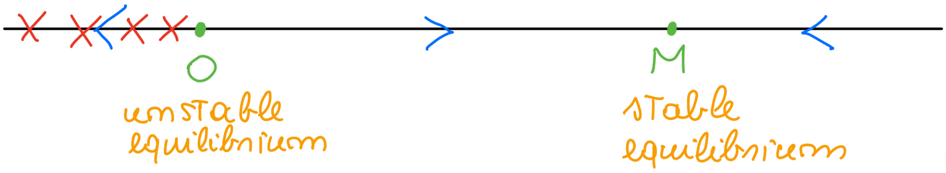

Figure 1.1. Phase line for $P' = kP(M-P)$.

If desired, it is possible to simplify the expression in the middle summand via the trigonometric identity $\tan^2(y)+1 = \cos^{-2}(y)$, which gives $\cos y = (\tan^2(y)+1)^{-1/2}$ and thus $\sin y = \sqrt{1-(\tan^2(y)+1)^{-1}}$, from which

$$\begin{aligned}\sin(2\arctan(x+1)) &= 2\sin(\arctan(x+1))\cos(\arctan(x+1))\\ &= 2\sqrt{1-\frac{1}{(x+1)^2+1}}\cdot\frac{1}{\sqrt{(x+1)^2+1}} = \frac{2(x+1)}{((x+1)^2+1)^2}\,.\end{aligned}$$

1.4.2. An application: the logistic population model. The *logistic equation* is a differential equation arising from a model proposed to describe the evolution of the population of a given species in a given environment. The physical assumption underpinning the model is that the rate of change of the population is proportional both to the population itself and to the *remaining space*, namely the difference between the maximum capacity allowed by the resources of the environments and the current population. Calling $M > 0$ such a maximum capacity, and letting $k > 0$ be the proportionality constant, the resulting differential equation satisfied by the function $P(t)$, indicating the population at time t, is

$$P'(t) = kP(t)(M - P(t))\,.$$

Before proceeding with finding explicit analytic formulas for the solutions, we perform a qualitative analysis by means of the phase line. The logistic equation is autonomous, and the zeros of the polynomial $kP(M-P)$ (in the variable P) appearing on the right-hand side are 0 and M. As the graph of the polynomial is a concave parabola, we have that it is positive in $(0, M)$ and negative in $(-\infty, 0)$ and $(M, +\infty)$, so that the phase line is as depicted in Figure 1.1.

As the model describes a population, we disregard any initial condition $P(0) = p_0 < 0$. We have two equilibrium states for the system, namely 0 and M, with 0 unstable and M stable: the population, if it starts with a non-zero number of individuals, approaches asymptotically the maximum capacity M.

Let us now solve the equation explicitly. We divide both sides of the equation by $P(t)(M - P(t))$ and integrate, so as to obtain

$$\int \frac{P'(t)}{P(t)(M-P(t))}\,\mathrm{d}t = kt + C\ , \quad C \in \mathbb{R}. \tag{1.4.3}$$

For the left-hand side, we need to evaluate

$$\frac{1}{P(M-P)}\,\mathrm{d}P\ ,$$

which is done via partial fraction decomposition. We write

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P} = \frac{A(M-P)+BP}{P(M-P)} = \frac{(B-A)P+AM}{P(M-P)}\ ,$$

so that we must have

$$\begin{cases} B - A = 0 \\ AM = 1 \end{cases}\ ,$$

that is, $A = B = M^{-1}$. Hence we get

$$\int \frac{1}{P(M-P)}\,\mathrm{d}P = \frac{1}{M}\Big(\int \frac{1}{P}\,\mathrm{d}P + \int \frac{1}{M-P}\,\mathrm{d}P\Big) = \frac{1}{M}(\log|P| - \log|M-P|) + C\ , \quad C \in \mathbb{R}.$$

From (1.4.3) we obtain

$$\frac{1}{M}\log\left|\frac{P(t)}{M-P(t)}\right| = kt + C\ ,$$

whence, calling $D = \pm e^{MC}$,

$$\frac{P(t)}{M-P(t)} = De^{kMt} \implies P(t) = D(M-P(t))e^{kMt} \implies P(t) = \frac{DMe^{kMt}}{1+De^{kMt}} = \frac{M}{1+D^{-1}e^{-kMt}}\ .$$

We observe that, as both k and M are positive, $\lim_{t\to+\infty} e^{-kMt} = 0$, so that

$$\lim_{t\to+\infty} P(t) = M$$

irrespective of the constant D^{-1} (whose explicit value can be determined in the presence of an initial condition for $P(t)$). This clearly matches what we had found out earlier by means of the phase line. Notice, however, that the explicit analytic formula for the solution tells us something more: $P(t)$ tends towards the equilibrium solution *at an exponential rate*, that is, with the same speed as e^{-kMt} decays to 0.

1.5. Improper integrals

In Calculus I, the definite integral

$$\int_a^b f(x)\,\mathrm{d}x$$

of a function $f\colon [a,b] \to \mathbb{R}$ has beed defined as the limit of Riemann sums $\sum_{i=1}^n f(x_i^*)(x_i - x_{i-1})$ where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, $x_i^* \in (x_{i-1}, x_i]$ for all $1 \le i \le n$ (or $x_i^* \in [x_{i-1}, x_i)$ for all $1 \le i \le n$), and the limit is taken as the size $\max_{i=1,\dots,n} x_i - x_{i-1}$ tends to 0. For a review, see Section C.1 in the Appendix.

The limit defining the definite integral may or may not exist. It has been established in Calculus I that it exists in $\mathbb{R}$ whenever f is continuous on $[a,b]$. Now it is possible to extend the definition of the integral to encompass more general cases, chiefly the following four:

(1) the function f is continuous on $[a,b)$ but has a discontinuity at b, or is continuous on $(a,b]$ but has a discontinuity at a, where by a discontinuity, by slight abuse of terminology, we also include the possibility that f is not defined at a or b;
(2) the function f is continuous on $[a,b]$ except at an internal point $c \in (a,b)$, with the possibility that f is not defined at c;

(3) the domain of definition of f is a half-line $[a,+\infty)$ or $(-\infty,b]$, and we want to make sense of the expressions

$$\int_a^{+\infty} f(x)\,\mathrm{d}x\ , \quad \int_{-\infty}^{b} f(x)\,\mathrm{d}x\ ;$$

(4) the domain of definition of f is the whole line $\mathbb{R}$, and we want to make sense of the integral

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x\ .$$

In all the previous cases, the integrals

$$\int_a^b f(x)\,\mathrm{d}x\ , \quad \int_a^{+\infty} f(x)\,\mathrm{d}x\ , \quad \int_{-\infty}^{b} f(x)\,\mathrm{d}x\ , \quad \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$$

are called **improper integrals**, as they cannot be defined as *proper* definite integrals as seen in Calculus I.

We shall extend the notion of integral to all the previous cases, dealing with each of them separately.

The function has a discontinuity at a boundary point of the interval of definition. Let's suppose that $f\colon [a,b]\to\mathbb{R}$ is continuous on $[a,b)$ and has a discontinuity at b; potentially, it is not defined at b. We define

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-}\int_a^t f(x)\,\mathrm{d}x$$

if the limit exists, potentially equal to $\pm\infty$, and say that the improper integral **converges** if the limit is finite, or **diverges** if the limit is infinite or does not exist. Similarly, if f is continuous on $(a,b]$ but has a discontinuity at a (potentially f is not defined at a), we define

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+}\int_t^b f(x)\,\mathrm{d}x$$

if the limit exists, potentially equal to $\pm\infty$, and say that the improper integral **converges** if the limit is finite, or **diverges** if the limit is infinite or does not exist.

Observe that the integrals $\int_a^t f(x)\,\mathrm{d}x$ are well-defined definite integrals in the first case for any $t\in[a,b)$, as f is continuous on $[a,b)$, and so are the integrals $\int_t^b f(x)\,\mathrm{d}x$ in the second case for any $t\in(a,b]$, as f is continuous on $(a,b]$.

Example 1.5.1. Let's evaluate the integral

$$\int_0^2 x^2\log x\,\mathrm{d}x\ .$$

The function $x^2\log x$ is not defined[7] for $x=0$, hence the integral is improper. Applying the definition, we have

$$\begin{aligned}\int_0^2 x^2\log x &= \lim_{t\to 0^+}\int_t^2 x^2\log x\,\mathrm{d}x = \lim_{t\to 0^+}\frac{x^3}{3}\log x\Big|_{x=t}^{x=2} - \frac{1}{3}\int_t^2 x^2\,\mathrm{d}x \\ &= \lim_{t\to 0^+}\frac{1}{3}(8\log 2 - t^3\log t) - \frac{1}{9}x^3\Big|_{x=t}^{x=2} = \frac{8\log 2}{3} - \frac{8}{9} + \lim_{t\to 0^+}\frac{t^3}{9} - \frac{t^3\log t}{3} \\ &= \frac{8\log 2}{3} - \frac{8}{9}\ ,\end{aligned}$$

where the limit

$$\lim_{x\to 0^+} t^3\log t = 0$$

[7]Observe that $x^2\log x$ can be extended by continuity to $x=0$, by setting it to be equal to 0 at 0: see Example 3.1.8.

can be computed with l'Hôpital's rule as in Example 3.1.8.

EXAMPLE 1.5.2. We would like to evaluate

$$\int_0^3 \frac{1}{(x-3)^2}\,\mathrm{d}x\,.$$

The function $(x-3)^{-2}$ is not defined at $x=3$, hence this is an improper integral. According to the definition, we have

$$\int_0^3 \frac{1}{(x-3)^2}\,\mathrm{d}x = \lim_{t\to 3^-}\int_0^t \frac{1}{(x-3)^2}\,\mathrm{d}x = \lim_{t\to 3^-} -(x-3)^{-1}\Big|_{x=0}^{x=t} = \lim_{t\to 3^-} -\frac{1}{t-3}-\frac{1}{3} = +\infty\,.$$

The function has a discontinuity at an internal point of the interval of definition. Suppose f is defined on an interval $[a,b]$ except for one point $c\in(a,b)$, or else that f is defined over $[a,b]$ and is continuous over $[a,b]$ except for a point of discontinuity $c\in(a,b)$. We then define

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to c^-}\int_a^t f(x)\,\mathrm{d}x + \lim_{t\to c^+}\int_t^b f(x)\,\mathrm{d}x$$

if both limit exist, with at least one of them finite, and say that the improper integral **converges** if both limits are finite, or **diverges** if at least one of them is infinite or at least one of the two limits does not exist.

EXAMPLE 1.5.3. We compute

$$\int_{-1}^1 \frac{1}{\sqrt{|x|}}\,\mathrm{d}x$$

As the function $\frac{1}{\sqrt{|x|}}$ is not defined at the point $x=0$, this is an improper integral with a singularity inside the domain of integration. We apply the definition and write

$$\int_{-1}^1 \frac{1}{\sqrt{|x|}}\,\mathrm{d}x = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{|x|}}\,\mathrm{d}x + \lim_{t\to 0^-}\int_{-1}^t \frac{1}{\sqrt{|x|}}\,\mathrm{d}x = 2\lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x\,,$$

where the last step follows from the fact that

$$\int_t^1 \frac{1}{\sqrt{|x|}}\,\mathrm{d}x = \int_{-1}^{-t} \frac{1}{\sqrt{|x|}}\,\mathrm{d}x$$

for any $t\in(0,1)$. Thus

$$\int_{-1}^1 \frac{1}{\sqrt{|x|}}\,\mathrm{d}x = 4\lim_{t\to 0^+}\sqrt{x}\Big|_{x=t}^{x=1} = 4\,.$$

The integral is extended over a half-line. Suppose f is a function defined and continuous over the closed half-line $[a,+\infty)$, where $a\in\mathbb{R}$. We define the improper integral

$$\int_a^{+\infty} f(x)\,\mathrm{d}x = \lim_{t\to+\infty}\int_a^t f(x)\,\mathrm{d}x$$

if the limit exists, and say that the impropert integral **converges** if the limit is finite, and **diverges** if the limit is infinite.

EXAMPLE 1.5.4. We evaluate

$$\int_1^{+\infty} \frac{1}{x}\,\mathrm{d}x\,.$$

It is an improper integral as the domain of integration is unbounded. Using the definition,

$$\int_1^{+\infty} \frac{1}{x}\,\mathrm{d}x = \lim_{t\to+\infty}\int_1^t \frac{1}{x}\,\mathrm{d}x = \lim_{t\to+\infty}\log x\Big|_{x=1}^{x=t} = \lim_{t\to+\infty}\log t = +\infty\,,$$

hence the integral diverges.

The integral is extended over the whole line. The last case is when we have a function f defined and continuous over $\mathbb{R}$, and we would like to make sense of the integral of f over $\mathbb{R}$ itself. We define the improper integral

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = \lim_{t\to+\infty}\int_0^t f(x)\,\mathrm{d}x + \lim_{t\to-\infty}\int_{-\infty}^0 f(x)\,\mathrm{d}x$$

if both limits exist, with at least one of them finite, and say that the improper integral **converges** if both of them are finite, and **diverges** if one of them is finite and the other one is infinite.

Remark 1.5.5. In the definition, we choose for simplicity the origin 0 as a starting point for the two definite integrals, but it would be equivalent to choose any other point $c \in \mathbb{R}$. Indeed, we have

$$\int_c^t f(x)\,\mathrm{d}x = \int_c^0 f(x)\,\mathrm{d}x + \int_0^t f(x)\,\mathrm{d}x\ , \quad \int_t^c f(x)\,\mathrm{d}x = \int_t^0 f(x)\,\mathrm{d}x + \int_0^c f(x)\,\mathrm{d}x$$

by the additivity property of integrals over disjoint sub-intervals. Thus

$$\begin{aligned}\lim_{t\to+\infty}\int_c^t f(x)\,\mathrm{d}x + \lim_{t\to-\infty}\int_t^c f(x)\,\mathrm{d}x &= \int_c^0 f(x)\,\mathrm{d}x + \lim_{t\to+\infty}\int_0^t f(x)\,\mathrm{d}x + \Big(\lim_{t\to-\infty}\int_t^0 f(x)\,\mathrm{d}x\Big) \\ &\quad + \int_0^c f(x)\,\mathrm{d}x \\ &= \lim_{t\to+\infty}\int_0^t f(x)\,\mathrm{d}x + \lim_{t\to-\infty}\int_t^0 f(x)\,\mathrm{d}x\ .\end{aligned}$$

Example 1.5.6. We would like to compute

$$\int_{-\infty}^{+\infty} e^{-3x}\,\mathrm{d}x\ ,$$

which is an improper integral as we are integrating over the whole real line. According to the definition,

$$\int_{-\infty}^{+\infty} e^{-3x}\,\mathrm{d}x = \lim_{t\to+\infty}\int_0^t e^{-3x}\,\mathrm{d}x + \lim_{t\to-\infty}\int_t^0 e^{-3x}\,\mathrm{d}x\ .$$

We compute the two limits separately. For the first one,

$$\lim_{t\to+\infty}\int_0^t e^{-3x}\,\mathrm{d}x = \lim_{t\to+\infty} -\frac{1}{3}e^{-3x}\Big|_{x=0}^{x=t} = \lim_{t\to+\infty}\frac{1}{3}(1-e^{-3t}) = \frac{1}{3}\ .$$

For the second one,

$$\lim_{t\to-\infty}\int_t^1 e^{-3x}\,\mathrm{d}x = \lim_{t\to-\infty} -\frac{1}{3}e^{-3x}\Big|_{x=t}^{x=0} = \lim_{t\to-\infty} -\frac{1}{3}+\frac{1}{3}e^{-3x} = +\infty\ ;$$

therefore, the integral $\int_{-\infty}^{+\infty}\ \mathrm{d}x$ diverges.

CHAPTER 2

Ordinary differential equations

Differential equations model a wealth of physical phenomena, and find applications to several other sciences such as economics, biology and engineering. This chapter is devoted to the development of a few basic themes concerning what are known as *ordinary differential equations*, namely those involving functions of a single variable. Since we will always confine ourselves to those, henceforth a differential equation is intended to be synonymous with ordinary differential equation.

2.1. Definitions and examples

A differential equation is an equation involving a function $y(x)$ and its higher-order derivatives $y', y'', \ldots$. The function y represents the unknown of the equation, and solving a differential equation means finding the unknown function f verifying the given equation.

Let us first look at some explicit examples, before proceeding with the general definitions.

Example 2.1.1. Let's consider the differential equation

$$y'(x) = x(x+1)\ . \tag{2.1.1}$$

Solving it means finding all possible functions y whose derivative equals the function $x(x+1)$. This clearly amounts to computing the indefinite integral

$$\int x(x+1)\,\mathrm{d}x = \int x^2 + x\,\mathrm{d}x = \frac{x^3}{3} + \frac{x^2}{2} + C\ , \quad C \in \mathbb{R}.$$

Hence, a function $y(x)$ solves the differential equation (2.1.1) if and only if it is of the form

$$y(x) = \frac{x^3}{3} + \frac{x^2}{2} + C$$

for some $C \in \mathbb{R}$. We have thus solved the equation completely.

The previous was an elementary example of a differential equation, in which the complete set of solutions can be found via a direct integration. Let us look at more involved examples of differential equations.

Example 2.1.2. The following are all examples of differential equations:

- $y'(x) = y(x)$;
- $y'(x) = e^{y(x)}$;
- $y'(x) = xy(x)$;
- $y''(x) + 2y'(x) + y(x) = 4\sin x$;
- $xy^2(x) - x^3 - y^{(3)}(x) = 0$.

In order to solve differential equations such as those appearing in the previous example, we need to develop more sophisticated tools; this is the main goal of the present chapter.

Observe that the first three examples in Example 2.1.2 are differential equations in which only the function y and its first derivative y' appear; by contrast, in the fourth example we also have the second derivative y'' appearing, and in the last one we have the third derivative $y^{(3)}$. We say that a differential equation has *order* k, where $k \geq 1$ is an integer, if the highest-order derivative of the unknown appearing in the equation is the k-th derivative. Hence, for instance, the first three equations in Example 2.1.2 are of first order, the fourth one is of second order, and the last one is of third order.

Standing assumption. We shall be almost exclusively concerned with first order differential equations; therefore, unless otherwise specified, a differential equation shall from now on be tacitly intended to be of first order.

Furthermore, we will always treat the case of differential equations *in normal form*, namely those appearing in the form $y'(x) = F(x, y(x))$ for some function F, where thus the right-hand side does not involve the derivative of the unknown function.

For instance, the first three equations in Example 2.1.2 are expressed in normal form.

Let us now give the abstract definition.

Definition 2.1.3 (Differential equation, solutions). A **differential equation** is an equation of the form

$$y'(x) = F(x, y(x)) \ , \tag{2.1.2}$$

where $F\colon (a, b) \times (c, d) \to \mathbb{R}$ is a function of two variables and a, b, c, d are real numbers with $a < b$ and $c < d$; possibly, a or c equals $-\infty$ and b or d equals $+\infty$. A **solution** of the differential equation $y'(x) = F(x, y(x))$ is a differentiable function $y\colon (a, b) \to (c, d)$ such that, for every $x \in (a, b)$, the equality $y'(x) = F(x, y(x))$ holds.

Remark 2.1.4. Observe that, since the function F is assumed to take as input two real numbers, the first one of which has to lie in (a, b), and the second one in (c, d), it is meaningless to look for possible solutions to the equation in (2.1.2) defined outside of the interval (a, b), or taking values outside of the interval (c, d). On the other hand, if y is defined on (a, b) and takes values (c, d), then the expression $F(x, y(x))$ is indeed well-defined for any $x \in (a, b)$.

Remark 2.1.5. Most of the time, F shall actually admit as input any two real numbers, that is, it shall be defined over the whole product set $\mathbb{R} \times \mathbb{R}$. Nevertheless, it might not be possible to solve (2.1.2) for all values $x \in \mathbb{R}$, but just for x lying in some proper sub-interval of $\mathbb{R}$. We will see plenty of such cases.

Example 2.1.6. To see how the abstract definition matches the examples given thus far, let's understand what the function F from Definition 2.1.3 is in the first three cases of Example 2.1.2:

- $y'(x){=}y(x)$: here $F(x, y) = y$, it does not depend on the first variable x, and is defined over the whole product $\mathbb{R} \times \mathbb{R}$;
- $y'(x) = e^{y(x)}$: here $F(x, y) = e^y$, and again it doesn't depend on the first variable and is defined on $\mathbb{R} \times \mathbb{R}$;
- $y'(x) = xy(x)$: here $F(x, y) = xy$, it does depend on both variables, and it's defined on $\mathbb{R} \times \mathbb{R}$.

Example 2.1.7. Consider the differential equation

$$y'(x) = \sqrt{x^5 - 2}\sqrt{y^2(x) - 1} \ .$$

Here the function F is given by $F(x, y) = \sqrt{x^5 - 2}\sqrt{y^2 - 1}$; in order for it to be well-defined, we need to ensure that $x^5 - 2 \geq 0$ and $y^2 - 1 \geq 0$. In accordance with Definition 2.1.3, possible domains of definition of F are, for instance, $(2^{1/5}, +\infty) \times (1, +\infty)$ and $(2^{1/5}, +\infty) \times (-\infty, -1)$.

2.2. A celebrated example: Newton's second law of motion

A rich source of examples of differential equations is provided by *Newton's second law of motion*, which in concise terms postulates that

$$F = ma \ ,$$

that is, the acceleration a given object undergoes, at any moment of time, is proportional to the total external force acting upon it, via a proportionality constant given by the inverse of the object's mass. In general, the force F depends both on the position and the velocity of the object. The latter is the first derivative of the position function, and the acceleration is the second derivative; therefore, we are dealing with a second-order differential equation.

In conformity with our standing assumptions, we shall only treat examples in which there is a way to reduce matters to a first-order differential equation. A first case occurs when the force imparted on the object is constant in time, as in the following example.

Example 2.2.1. Let's suppose that an object is only subject to the gravitational force which, in the customary approximation when the object is close to the surface of the earth, can be assumed to be constant, directly proportional to the mass of the object via a proportionality constant equal to a certain value g. For simplicity, we assume to orient our reference frame so that $g > 0$. In this setup, calling $v(t)$ the velocity of the object at time t, Newton's second law reads

$$mv'(t) = mg\ , \quad \text{that is,} \quad v'(t) = g\ .$$

Determining the velocity function becomes a triviality at this point: by an elementary integration, the anti-derivatives of the constant function g are the functions of the form $gt + C$ as C varies in $\mathbb{R}$; therefore, we obtain the set of solutions

$$v(t) = gt + C\ , \quad C \in \mathbb{R}\ .$$

It is natural to wonder what accounts for the additional freedom we have in the choice of the constant C, or in other words, why it is physically sensible to expect that there is not a unique possible solution. If two objects with the same mass start, say at time $t = 0$, with different velocities $\mathbf{v}_0, \mathbf{v}_1 \in \mathbb{R}$, and are both subject to the gravitational force, the differential equation describing their velocity is the same for both; nevertheless, it is natural to expect the two velocity functions $v_0(t), v_1(t)$ to be different. Indeed, $v_0(t)$ must satisfy the following two conditions simultaneously:

- $v_0(0) = \mathbf{v}_0$, namely the initial velocity is $\mathbf{v}_0$;
- there exists $C \in \mathbb{R}$ such that $v_0(t) = gt + C$ for every t; this is given by solving the differential equation, as we have seen.

Combining the two conditions, we deduce that $\mathbf{v}_0 = v_0(0) = g \cdot 0 + C = C$, so that C is precisely equal to the initial velocity $\mathbf{v}_0$. Thus the velocity function for the first object is given by $v_0(t) = gt + \mathbf{v}_0$. Similarly, for the second object, the velocity function is $v_1(t) = gt + \mathbf{v}_1$, and since $\mathbf{v}_0 \neq \mathbf{v}_1$, we see indeed that the two velocity functions are different.

The outcome of such considerations is a manifestation of what is known as *Newton's determinism*: knowing the velocity of an object, which is only subject to the gravitational force, at a given instant of time, determines uniquely its velocity at any future (and past) instant.

In the previous example, we may ask what's the *asymptotic behaviour* of the velocity function $v(t) = gt + C$, namely, what is $\lim_{t\to+\infty} v(t)$ if it exists. Since we oriented the frame so that $g > 0$, we see that such limit always equals $+\infty$, regardless of C, and thus by what we said above, regardless of the initial velocity of the object. Hence, the velocity grows indefinitely. This clearly doesn't match experimental evidence, thereby showing the flaws of such a simplistic model, wherewith we tried to describe the motion of an object under the action of the gravitational field. Let us amend the model in the upcoming example.

Example 2.2.2 (Taking air resistance into account). The model previously described is inaccurate not only as it can't account for the fact that the velocity changes direction abruptly when a falling object hits the ground, but more subtly as it doesn't consider friction, namely the resistance the air opposes to the motion of any object.

Friction acts in the direction opposite to the motion. Consistently with our daily experience, we may assume that it acts proportionally to the velocity, so that the overall force acting upon the object is given by $F(t) = mg - kv(t)$ for some real number $k > 0$. Notice that now the force is not necessarily constant in time, but rather varies with the velocity. The new differential equation is thus

$$v'(t) = g - \frac{k}{m} v(t) . \tag{2.2.1}$$

We will learn how to solve this equation. For the moment, let's check that the function

$$v(t) = \frac{mg}{k} + Ce^{-kt/m} , \tag{2.2.2}$$

where C is an arbitrary real number, solves the equation. On the one hand, we compute

$$v'(t) = -\frac{Ck}{m} e^{-kt/m} ;$$

on the other, we have

$$g - \frac{k}{m} v(t) = g - \frac{k}{m}\frac{mg}{k} - \frac{Ck}{m} e^{-kt/m} = -\frac{Ck}{m} e^{-kt/m} ,$$

which equals what we obtained for $v'(t)$. Therefore, we do have a (infinite family of) solution. We will see that, conversely, every solution of (2.2.1) takes the form (2.2.2).

Let's now examine, as we did for the previous model, the limiting behaviour of the new velocity function. We see that $\lim_{t\to+\infty} v(t) = mg/k$ for every $C \in \mathbb{R}$, since both k and m are positive and thus $e^{-kt/m} \to 0$ as $t \to +\infty$. The value mg/k, to which the velocity eventually levels off, is called the *terminal velocity* of the object under consideration.

2.3. Separable differential equations

Up to this point, we were only able to solve explicitly, those differential equations presenting themselves in the form $y'(x) = f(x)$ for some function f; direct integration gives that solutions to the equation are precisely anti-derivatives of $f(x)$. There is a broader class of differential equations which turns out to be amenable to analytic solution by integration, either by direct integration or via the substitution method. Those are *separable differential equations*.

Definition 2.3.1 (Separable differential equation). We say that a differential equation is **separable** if it admits the form

$$y'(x) = g(y(x))f(x) \tag{2.3.1}$$

for some given functions f and g.

In other words, a differential equation is separable if it expresses the derivative of the unknown function $y(x)$ as a product of two factors, one of which only depends upon the independent variable x, whereas the other only depends on the unknown function $y(x)$ itself.

Example 2.3.2. The following differential equations are separable:

- $y'(x) = x^3 + 1$; here, in our notation of (2.3.1), $f(u) = u^3 + 1$, $g(u) = 1$;
- $y'(x) = y(x)(y(x) - 4)$; here, $f(u) = 1$, $g(u) = u(u - 4)$;
- $y'(x) = (\tan y(x))^2 e^x$; here, $f(u) = e^u$, $g(u) = (\tan u)^2$.

Here is an example of a differential equation which is not separable:

$$y'(x) = x^2 + y(x)^2 .$$

It is apparent that the right-hand side cannot be written as a product of two functions[1], one only depending on x and the other only depending on y.

[1] As it happens, it is possible to obtain such an expression, provided we work over an appropriate extension of the real numbers, namely the *complex numbers*. They will be the subject of Chapter **??**.

2.3.1. A general method to solve separable equations. In order to find a solution to a separable differential equation as in (2.3.1), we proceed as follows. Assuming, for the time being, that a solution $y(x)$ exists and that $g(y(x)) \neq 0$ for all x is a certain open sub-interval of $\mathbb{R}$, we can divide both sides of the equation by $g(y(x))$ and thus write, for all such x,

$$\frac{y'(x)}{g(y(x))} = f(x) \ .$$

Now, two functions are equal if and only if their indefinite integrals are the same; thus we may take the indefinite integral on both sides of the previous equation, thereby obtaining

$$\int \frac{y'(x)}{g(y(x))} \, \mathrm{d}x = \int f(x) \, \mathrm{d}x \ . \tag{2.3.2}$$

Let now F and G be anti-derivatives of f and $1/g$, respectively. By the Fundamental Theorem of Calculus (see Corollary C.3.3 in the Appendix), these always exist if f and g are continuous. Theorem 1.1.1 ensures that the composition $G \circ y$ is an anti-derivative of $y'(x)/g(y(x))$. Put differently, we may apply the substitution $u = y(x)$ in the integral on the left-hand side of (2.3.2), so that $\mathrm{d}u = y'(x)\mathrm{d}x$, and thus get

$$\int \frac{y'(x)}{g(y(x))} \, \mathrm{d}x = \int \frac{1}{g(u)} \, \mathrm{d}u = G(u) + C = G(y(x)) + C \ . \tag{2.3.3}$$

On the other hand, we readily have

$$\int f(x) \, \mathrm{d}x = F(x) + C \tag{2.3.4}$$

for the indefinite integral on the right-hand side of (2.3.2). Therefore, we derive from (2.3.2) that

$$G(y(x)) = F(x) + C \ , \quad C \in \mathbb{R}. \tag{2.3.5}$$

Observe that, in principle, the two constants appearing in (2.3.3) and (2.3.4) are different, and thus should be denoted by different symbols C_1, C_2. However, in the end we would get

$$G(y(x)) = F(x) + C_2 - C_1 \ , \quad C_1, C_2 \in \mathbb{R},$$

and we may rename $C = C_2 - C_1$, which is again a constant allowed to take any real value.

What we reached in (2.3.5) is called an *implicit form* for the solution to the differential equation (2.3.1), as the unknown function $y(x)$ appears as argument of the function G. If G is invertible, we may solve the previous equation for $y(x)$, thus arriving at the *explicit form*

$$y(x) = G^{-1}(F(x) + C) \ , \quad C \in \mathbb{R}.$$

We can check that the expression we reached is indeed a solution of the original equation. Using the chain rule for derivatives, together with the formula for the derivative of an inverse function (both of which are recalled in Proposition B.2.2 of the Appendix), we compute

$$y'(x) = (G^{-1})'(F(x) + C)F'(x) = \frac{1}{G'(G^{-1}(F(x) + C))} F'(x) \ .$$

Now we know that $F' = f$ and $G' = 1/g$, whence we derive

$$y'(x) = g(G^{-1}(F(x) + C))f(x) = g(y(x))f(x) \ ,$$

as desired.

2.3.2. The method in concrete examples. Before stating formally a theorem ensuring the validity of previous deductions under the appropriate assumptions, let's practice the method in a few examples.

NOTATION. It is common practice to omit the independent variable x from the notation, when it indicates the variable on which the unknown function y depends. Thus, for instance, we will typically write

$$y' = y \quad \text{and} \quad y' = xy$$

for the differential equations

$$y'(x) = y(x) \quad \text{and} \quad y'(x) = xy(x) \ ,$$

respectively.

EXAMPLE 2.3.3. Let's solve the separable equation $y' = y^2 + 1$. Here the right-hand side only depends on the function y itself, so the procedure indicated above dictates that we should divide both sides of the equation by $y^2 + 1$ (which, incidentally, never vanishes). We obtain

$$\frac{y'(x)}{y^2(x) + 1} = 1 \ .$$

We now integrate both sides of the latter equation with respect to the independent variable[2] x:

$$\int \frac{y'(x)}{y^2(x) + 1} \, \mathrm{d}x = \int 1 \, \mathrm{d}x \ . \tag{2.3.6}$$

For the right-hand side, we readily get

$$\int 1 \, \mathrm{d}x = x + C \ , \quad C \in \mathbb{R}.$$

As for the left-hand side, we apply the substitution $u = y(x)$, $\mathrm{d}u = y'(x)\mathrm{d}x$, and get

$$\int \frac{y'(x)}{y^2(x) + 1} \, \mathrm{d}x = \int \frac{1}{u^2 + 1} \, \mathrm{d}u = \arctan u + C = \arctan y(x) + C \ , \quad C \in \mathbb{R}.$$

Therefore, (2.3.6) translates into

$$\arctan y(x) = x + C \ , \quad C \in \mathbb{R}.$$

We can now find an explicit expression for $y(x)$ by applying the tangent function on both sides of the last displayed equality: we obtain

$$y(x) = \tan(x + C) \ , \quad C \in \mathbb{R}.$$

Let's check that, indeed, the previous is a solution to the original equation $y' = y^2 + 1$ for every $C \in \mathbb{R}$. We compute

$$y'(x) = \left(\frac{\sin(x + C)}{\cos(x + C)}\right)' = \frac{\cos^2(x + C) + \sin^2(x + C)}{\cos^2(x + C)} = \frac{1}{\cos^2(x + C)} \ ;$$

on the other hand,

$$y^2(x) + 1 = \tan^2(x + C) + 1 = \frac{\sin^2(x + C)}{\cos^2(x + C)} + 1 = \frac{\sin^2(x + C) + \cos^2(x + C)}{\cos^2(x + C)} = \frac{1}{\cos^2(x + C)} \ ,$$

which is indeed equal to the expression we have for $y'(x)$.

Observe that the solutions $y(x) = \tan(x + C)$ are not defined for every value of x; for each fixed $C \in \mathbb{R}$, the function $\tan(x + C)$ is only well defined on infinitely many open intervals. For instance, let's pick the solution with $C = 0$, namely $y(x) = \tan x$: it is defined over the open intervals

$$(-\pi/2 + k\pi, \pi/2 + k\pi) \ , \quad k \in \mathbb{Z}.$$

Compare this with Remark 2.1.5.

[2]Observe that integrating with respect to y makes no sense.

Example 2.3.4. Let's consider the separable differential equation

$$y' = ky \ ,$$

where k is a given real number. We apply the method described in Section 2.3.1, divide both sides of the equation by y, and integrate, obtaining

$$\int \frac{y'(x)}{y(x)} \, \mathrm{d}x = \int k \, \mathrm{d}x \ .$$

The substitution $u = y(x)$, $\mathrm{d}u = y'(x)\mathrm{d}x$ allows to evaluate

$$\int \frac{y'(x)}{y(x)} \, \mathrm{d}x = \int \frac{1}{u} \, \mathrm{d}u = \log|u| + C = \log|y(x)| + C \ , \quad C \in \mathbb{R}.$$

On the other hand, we clearly have

$$\int k \, \mathrm{d}x = kx + C \ , \quad C \in \mathbb{R},$$

as k is a constant. We have thus reached the equality

$$\log|y(x)| = kx + C \ , \quad C \in \mathbb{R},$$

from which we deduce

$$|y(x)| = e^{C+kx} \ ,$$

and thus

$$y(x) = \pm e^C e^{kx} \ , \quad C \in \mathbb{R}.$$

Now observe that, as C varies in $\mathbb{R}$, the factor $\pm e^C$ appearing in front of the exponential function e^{kx} in the last displayed equality can take on arbitrary *non-zero* values (an exponential function never vanishes). We may simply rename it as D, another constant, and so write

$$y(x) = De^{kx} \ , \quad D \in \mathbb{R} \setminus \{0\}. \tag{2.3.7}$$

Notice that, in order to divide by y in the first step of the method, we implicitly assumed that $y \neq 0$. What if we are interested in the solution of the differential equation $y' = ky$ having 0 as initial value, namely satisfying $y(0) = 0$? In this case, we simply observe that the constant function $y(x) = 0$ is also a solution to the equation $y' = ky$; we realize that this constant solution is captured by allowing the constant D in (2.3.7) to take the value 0. We shall return to this point later on, when addressing the notion of *equilibrium state* (cf. Section 2.4).

To summarize, the full set of solutions to the differential equation $y' = ky$ is given by

$$y(x) = De^{kx} \ , \quad D \in \mathbb{R}.$$

Notice that each such solution is well defined for every $x \in \mathbb{R}$.

A final check of the correctness of our results: computing the derivative of $y(x) = De^{kx}$ gives $y'(x) = kDe^{kx} = ky(x)$.

Example 2.3.5. Consider the separable differential equation

$$y' = ky^2 \ ,$$

where k is a given real number. Dividing both sides of the equation by y^2 and integrating, we get

$$\int \frac{y'(x)}{y^2(x)} \, \mathrm{d}x = \int k \, \mathrm{d}x \ ,$$

which, by the usual substitution $u = y(x)$ and the fact that $\int u^{-2} \, \mathrm{d}u = -u^{-1}$, results in

$$-\frac{1}{y(x)} = kx + C \ , \quad C \in \mathbb{R}.$$

Solving for $y(x)$, we obtain

$$y(x) = -\frac{1}{kx + C} \ , \quad C \in \mathbb{R}.$$

For each $C \in \mathbb{R}$, the corresponding solution is defined for all $x \neq -C/k$, hence on the two open half-lines $(-\infty, -C/k)$, $(C/k, +\infty)$.

2.3.3. Newton's law of cooling. Another example of a differential equation modelling a physical phenomenon is given by *Newton's law of cooling*, which describes the rate at which the temperature of a given object changes, as a function of the temperature itself and of the temperature of the surrounding environment. The assumption underpinning the law is that the rate of change of the temperature is proportional to the difference between the temperature of the object and the temperature of the environment. Let us denote by $T(t)$ the temperature of the object at time t; if the surrounding environment is much larger than the object under consideration, it is reasonable to assume that the temperature of the former is constant in time, say equal to a certain value T_0. With such notation in place and in light of the physical assumptions just discussed, the differential equation embodying Newton's law of cooling takes the form

$$T'(t) = k(T(t) - T_0) \ .$$

Experimental evidence shows that the temperature of the object increases if it is lower than T_0, and decreases if it is larger. Therefore, it is justified by the physical intuition to confine our attention to the case $k < 0$. At this point we prefer to write the proportionalty constant $-k$ for some $k > 0$; this shall facilitate the interpretation of the solutions we will get. We thus rewrite the equation as

$$T'(t) = -k(T(t) - T_0) \ . \tag{2.3.8}$$

Example 2.3.6. Suppose that a pot of boiling water, with initial temperature of 212° F, is placed in a room at temperature 50° F. Assume we know that the pot's temperature begins to change at a rate of 3° F per minute. Can we determine the proportionality constant k in this setting?

As a matter of fact, we can determine it without even attempting to solve the equation, but simply relying on our initial data. Indeed, we know that $T_0 = 50$ and $T(0) = 212$ (both measured in Fahrenheit). Finally, we know that $T'(0) = -3$ (measured in Fahrenheit per minute, the negative sign being due to the obvious fact that the pot cools down); plugging such values into (2.3.8) for $t = 0$, we readily obtain

$$-3 = -k(212 - 50) \implies k = 3/162 = 1/54 \ .$$

The units of k are $1/m$, with m standing for minutes.

Let us now solve the differential equation (2.3.8) explicitly. As is the case for all equations appearing in the present section, it is separable. Let's first assume that $T(t) - T_0 \neq 0$ for all t close enough to the initial time $t = 0$, so that we can divide both sides by $T(t) - T_0$. Integrating, we obtain

$$\int \frac{T'(t)}{T(t) - T_0} \, \mathrm{d}t = \int -k \, \mathrm{d}t = -kt + C \ , \quad C \in \mathbb{R}.$$

Since a primitive of the function $f(u) = 1/(u - T_0)$ is given by $\log |u - T_0|$, as shown by a trivial substitution shows, we deduce that

$$\log |T(t) - T_0| = -kt + C \ , \quad C \in \mathbb{R}.$$

Taking the exponential on both sides, removing the absolute value and taking the summand $-T_0$ to the other side, we get

$$T(t) = T_0 \pm e^{C - kt} \ ,$$

which we can rewrite as

$$T(t) = T_0 + De^{-kt} \ , \quad D \in \mathbb{R} \setminus \{0\}$$

by setting $D = \pm e^C$. Observe that, when $D = 0$, we would obtain the constant function $T(t) = T_0$, which is also a solution to (2.3.8), only that it was not captured by the previous computations as we needed to assume from the start $T(t) \neq T_0$.

We have reached the following conclusion: the solutions to the differential equation (2.3.8) for Newton's law of cooling are given by

$$T(t) = T_0 + De^{-kt}\ , \quad D \in \mathbb{R}. \tag{2.3.9}$$

EXAMPLE 2.3.7. Let us go back to the previous example, where we know that $T_0 = 50$, $T(0) = 212$ and we established that $k = 1/54$. To determine the precise expression for the temperature function in this case, we need to solve the so-called **initial value problem**

$$\begin{cases} T'(t) = -\frac{1}{54}(T(t) - 50) \\ T(0) = 212\ , \end{cases}$$

which consists of a differential equation together with an initial datum. We know from (2.3.9) that

$$T(t) = 50 + De^{-t/54}$$

for some $D \in \mathbb{R}$, whose value is uniquely determined by imposing the initial condition $T(0) = 212$. We get

$$212 = T(0) = 50 + De^{-0/54} = 50 + D \implies D = 162\ .$$

Finally, we reach the expression $T(t) = 50 + 162e^{-t/54}$ for the pot's temperature. Thus, for instance, we know that after 1 minute the temperature will be $T(1) = 50 + 162e^{-1/54}$ Fahrenheit, and after 10 minutes it will be $T(10) = 50 + 162e^{-5/27}$ Fahrenheit.

To conclude this section, we observe that the same differential equation describing Newton's law of cooling can be encountered in disparate contexts, such as in the following two examples.

EXAMPLE 2.3.8. A large corporation withdraws money from an account at a constant rate of r dollars per year, and the account earns interest at a rate of 6% compounded continuously. Let $B(t)$ be the balance (measured in dollars) in the account as a function of time t (measured in years). What's the differential equation satisfied by B?

The information we are given about the balance $B(t)$ regards its rate of change, or in mathematical terms, its derivative $B'(t)$. We now that, on the one hand, $B(t)$ increases as the account earns interest, with a rate corresponding to 6% of the balance itself; on the other hand $B(t)$ decreases as an effect of the money withdrawal, which happens at a constant rate r. Combining the two pieces of information, we deduce that

$$B'(t) = \frac{6}{100}B(t) - r\ ,$$

which we may also write in the form

$$B'(t) = \frac{3}{50}\left(B(t) - \frac{50}{3}r\right)\ ,$$

which is ostensibly the same as for Newton's law of cooling, except for the fact that the constant factor $\frac{3}{50}$ in front is positive.

EXAMPLE 2.3.9. A patient is hooked up to an IV and is given medicine at a rate of 85 mg/hr. The medicine has a half life of 6 hours. Let $Q(t)$ be the amount (measured in mg) of medicine in the patient's blood as a function of time t (measured in hours). What is the differential equation satisfied by Q?

Recall that the half life of a drug is the time it takes for the amount of the drug's active substance in the body to reduce by half. Suppose first we were in the case where the patient is not given any medicine, what would be the differential equation satisfied by Q in this simplified case? We know that $Q(6) = \frac{1}{2}Q(0)$, namely the quantity of medicine in the blood after 6 hours is half the original quantity. Iterating, we deduce that after 12 hours the amount of medicine in the blood is $1/2 \cdot 1/2 = 1/4$ of the original one, that is, $Q(12) = Q(2 \cdot 6) = 2^{-2}Q(0)$. We can iterate this deduction as many times as we like: for every integer $n \geq 1$, we have $Q(6n) = 2^{-n}Q(0)$. It is natural to assume that this equality can be actually extended in continuous time, so that

$Q(6t) = 2^{-t}Q(0)$ for any real number t. We may clearly rewrite the previous, by an elementary change of variable, as $Q(t) = 2^{-t/6}Q(0)$. What is now the differential equation satisfied by Q? To find it from the explicit expression of Q, we compute its derivative and try to express it as a function of Q itself (and possibly of the time t as well). We compute

$$Q'(t) = -\frac{\log 2}{6} Q(0) 2^{-t/6} ,$$

and we thus realize that

$$Q'(t) = -\frac{\log 2}{6} Q(t) .$$

If we now take into account that the patient is hooked up to an IV, and is given the medicine at a rate of 85 mg/hr, this means that the rate of change of the medicine in the patient's blood is also affected by this constant increase. All in all, we infer that the differential equation satisfied by Q is

$$Q'(t) = 85 - \frac{\log 2}{6} Q(t) = -\frac{\log 2}{6}\left(Q(t) - \frac{6 \cdot 85}{\log 2}\right) ,$$

which is again of the same form as the equation describing Newton's law of cooling.

2.4. Equilibrium states and the qualitative analysis of solutions

As we have already alluded to in previous sections, differential equations inherently describe the infinitesimal law of evolution of a given quantity; in other words, they provide information about its short-term evolution. The typical endeavour is, predictably, to understand the long-term, or asymptotic behaviour of the solutions. This represents a minor challenge when explicit analytic formulas for the solutions are available, which has hitherto been always the case for us. Most frequently though, we lack such explicit formulas. A classical example is the so-called *three-body problem* in celestial mechanics: it is provably true that solutions to the differential equations describing the relative motion of three (or more) objects undergoing mutual gravitational interaction cannot be expressed by any closed formula in terms of elementary functions. How is it then possible to elicit information about the long-term evolution, say, of the solar system? To this end, novel tools need to be developed and implemented. The purpose of this section is to introduce the notion of equilibrium states of differential equations, and examine how they prove to be useful in a *qualitative analysis* of the long-term behaviour of solutions to *autonomous* differential equations.

2.4.1. Autonomous equations, stable and unstable equilibrium states. To begin with, we define the last two mentioned concepts.

Definition 2.4.1 (Equilibrium state). Let $y'(x) = F(x, y(x))$ be a differential equation as in Definition 2.1.3. An **equilibrium state** of the differential equation is a constant solution $y : \mathbb{R} \to \mathbb{R}$ to the equation: there exists some $C \in \mathbb{R}$ such that $y(x) = C$ for all $x \in \mathbb{R}$.

Equilibrium states are also called **equilibrium solutions**. For a function $y : \mathbb{R} \to \mathbb{R}$ to be an equilibrium state of $y'(x) = F(x, y(x))$ with constant value C, it is necessary and sufficient that $F(x, C) = 0$ for every $x \in (a, b)$ (cf. the notation used in Definition 2.1.3); indeed, if $y(x) = C$ for every x, then $y' = 0$ and thus

$$y'(x) = F(x, y(x)) \Longleftrightarrow 0 = F(x, C) .$$

As announced, we shall be concerned with equilibrium states of autonomous differential equations.

Definition 2.4.2 (Autonomous differential equation). An **autonomous differential equation** is a differential equation of the form

$$y'(x) = f(y(x))$$

for some function f.

Equivalently, referring to the notation in Definition 2.3.1, a differential equation is autonomous if and only if it is separable with the function f constantly equal[3] to 1.

Given an autonomous differential equation $y'(x) = f(y(x))$, a constant function $y(x) = C$ is an equilibrium state of it if and only if $f(C) = 0$; this follows directly from our previous considerations in the general case. Therefore, finding equilibrium states for the differential equation $y' = f(y)$ amounts to finding the zeros of the function f.

Example 2.4.3. Consider the autonomous equation $y'(x) = y^2(x) - 1$. The function f in this case is $f(y) = y^2 - 1$, and vanishes at the points 1 and -1. Thus, the equilibrium states of the equation are the constant functions $y(x) = 1$ and $y(x) = -1$.

Suppose given an autonomous equation $y' = f(y)$. Understanding the long-term evolution of equilibrium states is a trivial matter: they are precisely those solutions which never change in time. What about all other possible solutions to the equation? In order to gain some understanding of those, we distinguish between different types of equilibrium states. Henceforth, by a little abuse of terminology, *the constant value taken by an equilibrium state is also referred to as equilibrium state.*

Definition 2.4.4 (Stable and unstable equilibrium states). An equilibrium state $y_0 \in \mathbb{R}$ of an autonomous differential equation $y' = f(y)$ is called **stable** if there exists $\varepsilon > 0$ such that $f(y) < 0$ for all $y \in (y_0, y_0 + \varepsilon)$ and $f(y) > 0$ for all $y \in (y_0 - \varepsilon, y_0)$. It is called **unstable** if here exists $\varepsilon > 0$ such that $f(y) > 0$ for all $y \in (y_0, y_0 + \varepsilon)$ and $f(y) < 0$ for all $y \in (y_0 - \varepsilon, y_0)$.

By definition, equilibrium states of $y' = f(y)$ are points $y_0 \in \mathbb{R}$ such that $f(y_0) = 0$. The distinction between stable and unstable equilibrium states[4] invokes the behaviour of the function f for values of y close to y_0.

Example 2.4.5. Let us go back to the previous example $y' = y^2 - 1$, where we have the equilibrium states $1, -1$. To determine the nature of such equilibrium states, we need to examine the sign of the function f in the various intervals $(-\infty, -1)$, $(-1, 1)$, $(1, +\infty)$. The graph of the function f is a convex parabla, and we readily infer that $f(y) > 0$ for all $y \in (1, +\infty)$ and all $y \in (-\infty, -1)$, whereas $f(y) < 0$ for all $y \in (-1, 1)$. Therefore, -1 is a stable equilibrium and 1 is an unstable equilibrium.

Let's solve the equation explicitly; in so doing, we shall understand the reason underlying the terminology of stable and unstable. We have

$$\int \frac{y'(x)}{y^2(x) - 1}\, \mathrm{d}x = \int 1\, \mathrm{d}x\ . \tag{2.4.1}$$

We shall learn how to compute integrals of the form $\int \frac{1}{u^2-1}\, \mathrm{d}u$ in due course, via the method of *partial fraction decomposition*; for the moment, we just observe that we can write

$$\frac{1}{u^2 - 1} = -\frac{1}{2}\left(\frac{1}{u+1} - \frac{1}{u-1}\right),$$

so that, via the substitution $u = y(x)$,

$$\begin{aligned} \int \frac{y'(x)}{y^2(x) - 1}\, \mathrm{d}x = \int \frac{1}{u^2 - 1}\, \mathrm{d}u &= -\frac{1}{2}\left(\int \frac{1}{u+1}\, \mathrm{d}u - \int \frac{1}{u-1}\, \mathrm{d}u\right) \\ &= -\frac{1}{2}(\log|u+1| - \log|u-1|) + C \\ &= -\frac{1}{2}\log\left|\frac{y(x)+1}{y(x)-1}\right| + C\ . \end{aligned}$$

[3] Beware of the minor conflict of notation: in Definition 2.4.2 the function f plays the role of the function g in Definition 2.3.1.

[4] Observe that, as we shall see in examples, it may well happen that an equilibrium state is neither stable nor unstable.

Hence, from (2.4.1) we infer

$$-\frac{1}{2}\log\left|\frac{y(x)+1}{y(x)-1}\right| = x + C \implies \frac{y(x)+1}{y(x)-1} = \pm e^{-2(x+C)} \implies y(x) = \frac{\pm e^{-2(x+C)}+1}{\pm e^{-2(x+C)}-1} .$$

Setting $D = \pm e^{-2C}$, we obtain

$$y(x) = \frac{De^{-2x}+1}{De^{-2x}-1} , \tag{2.4.2}$$

where we now realize that D is actually allowed to be 0 as well, since the constant function -1 is a solution to the equation. Observe that the solution is not always well defined over the whole real line: when $D > 0$, we need the condition $x \neq \log D/2$, so that the solution is only well defined on $(-\infty, \log D/2)$ and $(\log D/2, +\infty)$.

Now, given an initial condition $y(0) = y_0$, let's determine the corresponding value of D: plugging in y_0 for $x = 0$ in (2.4.2), we have

$$y_0 = \frac{D-1}{D+1} \implies D = \frac{1+y_0}{1-y_0} ,$$

and thus the corresponding solution reads

$$y(x) = \frac{\frac{1+y_0}{1-y_0}e^{-2x}+1}{\frac{1+y_0}{1-y_0}e^{-2x}-1} . \tag{2.4.3}$$

We want to examine the asymptotic behaviour of the solution with initial condition y_0, where we might assume $y_0 \neq 1$ so that the expression (2.4.3) is well defined[5]. Some care is needed in what we mean by asymptotic behaviour. Indeed, if $\log D < 0$, that is, if $y_0 < 1$, then there is a solution defined for all times $x \geq 0$, so that it makes sense to examine $\lim_{x\to+\infty} y(x)$; however, if $\log D > 0$, then the solution starting at $x = 0$ is only well defined up to time $\log D/2$, so it only makes sense to examine $\lim_{x\to(\log D/2)^-} y(x)$. We compute

$$\begin{cases} \lim\limits_{x\to+\infty} y(x) = -1 & \text{if } y_0 < 1 \\ \lim\limits_{x\to(\log D/2)^-} y(x) = +\infty & \text{if } y_0 > 1 \end{cases} .$$

Therefore, we deduce that, whenever we start with values of y_0 close to the stable equilibrium -1, the solution approaches asymptotically the equilibrium itself. On the other hand, if we start with values of y_0 close to the unstable equilibrium 1, then the solution distances itself asymptotically from the equilibrium, approaching either $+\infty$ or the stable equilibrium -1.

This phenomenon is general, as we are going to see shortly, and explains the terminology: perturbing the initial condition around a stable equilibrium produces solutions *coming back* asymptotically to the equilibrium itself, while perturbing the initial condition around an unstable equilibrium gives rise to solutions *veering away* asymptotically from the equilibrium.

2.4.2. Existence and uniqueness of solutions. In order to proceed with our discussion, it is informative to state a general result about existence and uniqueness of solutions to differential equations with given initial conditions. Conforming to the setup of the current section, we confine ourselves to the autonomous case, though the theorem admits a much more general formulation.

To begin with, let us call a *Cauchy problem* the data of a differential equation[6] and an initial condition:

$$\begin{cases} y'(x) = f(y(x)) \\ y(x_0) = y_0 \end{cases} ,$$

where $f\colon (a,b) \to \mathbb{R}$ is a function, $x_0 \in \mathbb{R}$, $y_0 \in (a,b)$. A *maximal solution* to a Cauchy problem as above is, by definition, a differentiable function y defined on an open interval $I \subset \mathbb{R}$

[5]Note that for $y_0 = 1$ we know the asymptotic behaviour, as 1 is an equilibrium state.

[6]Once again, here we are confining ourselves to the autonomous case, but an entirely analogous definition can be formulated in full generality.

containing x_0, with values in (a,b), which fulfills $y'(x) = f(y(x))$ for every $x \in I$ (that is, it's a solution of the given differential equation), $y(x_0) = y_0$ and has the following property: there is no differentiable function $\tilde{y}$ with values in (a,b), defined over an open interval $\tilde{I}$ strictly containing I (and thus, in particular, x_0), and satisfying both $\tilde{y}'(x) = f(\tilde{y}(x))$ for every $x \in \tilde{I}$ and $\tilde{y}(x_0) = y_0$. In loose terms, a solution of a Cauchy problem is maximal if its domain of definition cannot be further enlarged.

THEOREM 2.4.6 (Cauchy-Lipschitz, a.k.a. Picard-Lindelöf). *Let $y'(x) = f(y(x))$ be an autonomous differential equation, where $f\colon (a,b) \to \mathbb{R}$ is a continuously differentiable function. Then, for every $x_0 \in \mathbb{R}$ and every $y_0 \in (a,b)$, there exists a unique maximal solution $y\colon I \to (a,b)$ to the Cauchy problem*

$$\begin{cases} y'(x) = f(y(x)) \\ y(x_0) = y_0 \end{cases} .$$

The proof of this theorem is decidedly beyond the scope of this course.

2.4.3. Phase lines and the qualitative analysis of solutions. With Theorem 2.4.6 at our disposal, we are in a position to fully understand, from a qualitative standpoint, the asymptotic behaviour of *every* solution of an autonomous differential equation.

Let thus $y'(x) = f(y(x))$ be an autonomous differential equation, and assume f is continuously differentiable. For simplicity of exposition, we shall assume that f is well-defined over the whole real line. Let $E \subset \mathbb{R}$ be the subset consisting of equilibrium states of the equation, equivalently

$$E = \{y \in \mathbb{R} : f(y) = 0\} .$$

For instance, if f is a polynomial function of y, then E is a finite set of points. We want to understand solutions to the equation with initial condition $y(0) = y_0$, where $y_0 \notin E$; for $y_0 \in E$, we already know that the corresponding unique solution is constant all the time, and there is not else much to say about its asymptotic behaviour.

It is a fact[7] that the complement of E in $\mathbb{R}$, namely the set $\{y \in \mathbb{R} : y \notin \mathbb{R}\}$, can be partitioned into a (possibly infinite) list of disjoint open intervals; furthermore, on each such interval (α, β), either $f(y) > 0$ for every $y \in (\alpha,\beta)$ or $f(y) < 0$ for every $y \in (\alpha, \beta)$.

The following theorem summarizes the behaviour of solutions of autonomous equations, depending on the sign of the function f at their initial value.

THEOREM 2.4.7. *Let $y'(x) = f(y(x))$ be an autonomous differential equation, where $f\colon (a,b) \to \mathbb{R}$ is a continuously differentiable function. Given $x_0 \in \mathbb{R}$ and $y_0 \in (a,b)$ consider the unique maximal solution $y\colon I = (c,d) \to \mathbb{R}$ to the Cauchy problem*

$$\begin{cases} y'(x) = f(y(x)) \\ y(x_0) = y_0 \end{cases} .$$

(1) If $f(y_0) = 0$, then $I = \mathbb{R}$ and $y(x) = y_0$ for every $x \in \mathbb{R}$.
(2) Suppose $f(y_0) > 0$, and let $(a_0, b_0) \subset (a,b)$ be the maximal[8] open interval containing y_0 on which $f > 0$. Then f is strictly increasing on I and

$$\lim_{x \to c^+} y(x) = a_0 , \quad \lim_{x \to d^-} y(x) = b_0 .$$

(3) Suppose $f(y_0) < 0$, and let $(a_0, b_0) \subset (a,b)$ be the maximal open interval containing y_0 on which $f < 0$. Then f is strictly decreasing on I and

$$\lim_{x \to c^+} y(x) = b_0 , \quad \lim_{x \to d^-} y(x) = a_0 .$$

[7]The proof requires elementary notions from the mathematical area of *topology*, which falls outside the scope of this course.

[8]The terminology is self-explanatory: there is no strictly bigger open interval contained in (a,b) with the same properties as (a_0, b_0).

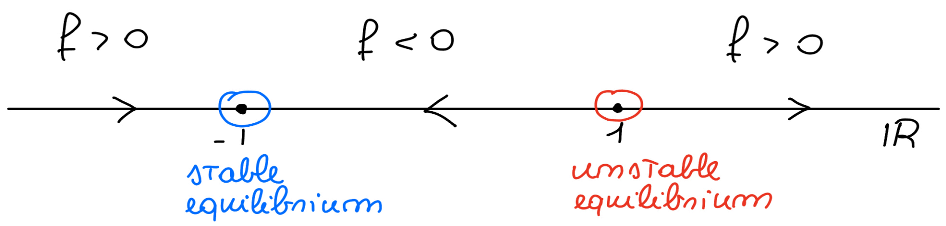

FIGURE 2.1. Phase line for $y' = y^2 - 1$.

The notions of strictly increasing/decreasing function are recalled in Definition A.2.1 of the Appendix.

REMARK 2.4.8. A couple of comments about Theorem 2.4.7 are in order.

- It can happen that f is defined over the whole $\mathbb{R}$, that is, $a = -\infty$ and $b = +\infty$. In this case, it may also happen that, in case (2) and (3) of the theorem, $a_0 = -\infty$ or $b_0 = +\infty$. See, for instance, Example 2.3.3.
- As opposed to case (1) of the theorem, it is by no means guaranteed that the solution in cases (2) and (3) is defined over the whole $\mathbb{R}$ (cf. Example 2.4.5). However, if it is, then the limits mentioned in the statement need to be appropriately interpreted: if $c = -\infty$, then $\lim_{x\to c^+}$ is meant to be $\lim_{x\to-\infty}$, and similarly if $d = +\infty$, $\lim_{x\to d^-}$ is meant to be $\lim_{x\to+\infty}$.

Graphically, all our previous considerations are best resumed with the help of a so-called *phase line*. Given an autonomous equation $y'(x) = f(y(x))$, we draw a horizontal line representing the set of real numbers $\mathbb{R}$. First, we identify the zeros of the function f, namely those $y \in \mathbb{R}$ for which $f(y) = 0$, and mark them on the line; these correspond to equilibrium states for the equation. Secondly, we study the sign of the function f, pinpointing those sub-intervals where $f > 0$ and those where $f < 0$. On each open sub-interval where $f > 0$, we draw a rightward-pointing arrow: this indicates that every solution taking values in the interval under consideration increases with time, thus pointing to the right on the line; similarly, on each sub-interval where $f < 0$, we draw a leftward-pointing arrow, indicating that every solution with initial value in such interval decreases with time, hence points to the left on the line. This procedure further enables us to detect immediately the nature of the equilibrium states: those with nearby arrows pointing towards them are stable, those for which nearby arrows point away from them are unstable, all others are neither stable nor unstable.

EXAMPLE 2.4.9. Figure 2.1 shows the phase line for the differential equation $y' = y^2 - 1$ examined in Examples 2.4.3 and 2.4.5. Clearly, the resulting qualitative analysis is consistent with the quantitative results obtained in Example 2.4.5 at the price of considerable more effort.

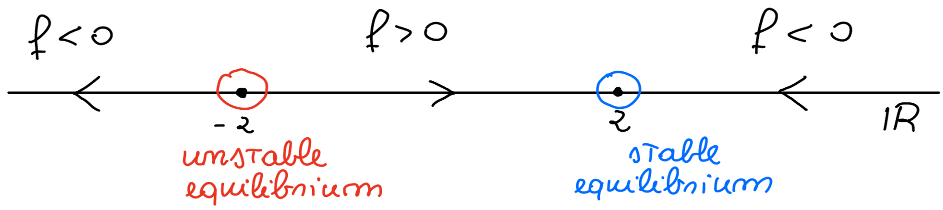

FIGURE 2.2. Phase line for $y' = 4 - y^2$.

2.4.4. Examples of phase lines. We shall now examine a few more examples, in order to understand in concrete circumstances how phase lines provide valuable qualitative information on the behaviour of solutions to autonomous differential equations.

EXAMPLE 2.4.10. Consider the autonomous equation $y' = 4 - y^2$. The function f to be studied in this case is the polynomial function $f(y) = 4 - y^2$; its graph is a concave parabola, with zeros 2 and -2. Furthermore, $f > 0$ on $(-2, 2)$ and $f(y) < 0$ on $(-\infty, 2)$ and $(2, \infty)$. Thus, the phase line in this example corresponds to what Figure 2.2 shows.

We observe that 2 is a stable equilibrium, whereas -2 is an unstable equlibrium. All solutions starting at a point in $(-2, 2)$ increase all the way up to 2, without ever touching it (nor do they ever touch -2 in the past). All solutions starting at a point in $(2, +\infty)$ increase all the way up to $+\infty$, and all solutions starting at a point in $(-\infty, -2)$ decrease all the way down to $-\infty$.

EXAMPLE 2.4.11 (Revisiting the fall of an object under gravitation and air resistance). Recall the differential equation

$$mv'(t) = mg - kv(t)$$

from Example 2.2.2, describing the motion of an object subject to gravitation and air resistance. Dividing by the mass m, we bring it into the usual form

$$v'(t) = g - \frac{k}{m}v(t).$$

The function f to be examined in this example is the linear polynomial $f(v) = g - \frac{k}{m}v$. Its graph is a straight line with negative slope and unique zero at mg/k; also, $f > 0$ on $(-\infty, mg/k)$ and $f < 0$ on $(mg/k, +\infty)$.The corresponding phase line is shown in Figure 2.3.

We see that the unique equilibrium mg/k, which in Example 2.2.2 we called terminal velocity, is stable: every solution to the equation converges to such value, asymptotically.

EXAMPLE 2.4.12 (Revisiting Newton's law of cooling). Recall the equation

$$T'(t) = -k(T(t) - T_0)$$

from Section 2.3.3, describing the temperature of an object placed in contact with a larger one at constant temperature T_0. The function f is $f(T) = -k(T - T_0)$, whose graph is again

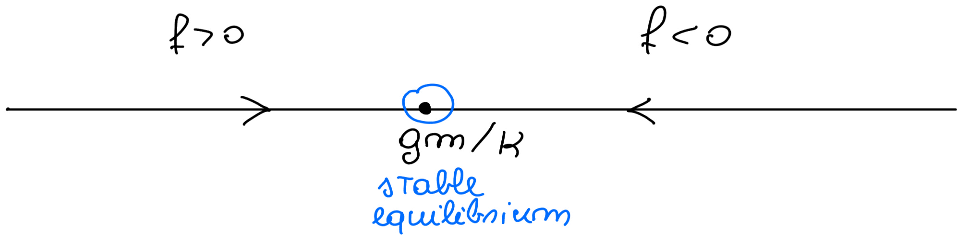

FIGURE 2.3. Phase line for $mv' = mg - kv$.

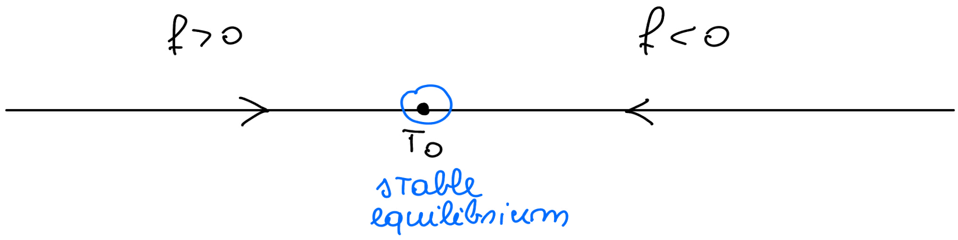

FIGURE 2.4. Phase line for $T' = -k(T - T_0)$.

a straight line with negative slope (recall that $k > 0$) and unique zero T_0. Also, $f > 0$ on $(-\infty, T_0)$ and $f < 0$ on $(T_0, +\infty)$. The phase line is shown in Figure 2.4.

As in the previous example, the unique equilibrium T_0 is stable; as seen in Section 2.3.3 by solving the equation explicitly, every solution approaches asymptotically the ambient temperature T_0.

CHAPTER 3

Indeterminate forms and growth rates of functions

3.1. Indeterminate forms and l'Hôpital's rule

One of the core topics the Calculus I course centered on is the notion of limits and continuity for real-valued functions of a real variable. The most direct way to compute limits is by *direct substitution*, which consists in using continuity of the functions involved and plugging in their value at their limit point. This is how you readily compute, for instance, that

$$\lim_{x\to 0} \frac{2x+1}{3x+2} = \frac{1}{2} ;$$

indeed, the function $f\colon x \mapsto \frac{2x+1}{3x+2}$, defined for all real $x \neq -2/3$, is continuous over its entire domain of definition, which includes the limit point 0, and thus

$$\lim_{x\to 0} f(x) = f(0) = \frac{1}{2} .$$

There are situations, however, where direct substitution is not applicable to compute a given limit

$$\lim_{x\to a} f(x) ,$$

either because the considered function is not defined at the point a, or because despite being defined at a it is not continuous at a, or else because $a = \pm\infty$. Frequently, in such cases, an attempt at "direct substitution" leads to an indeterminate form.

Definition 3.1.1 (Indeterminate form). Let $f, g\colon (a,b) \to \mathbb{R}$ be two functions defined over an interval $(a,b) \subset \mathbb{R}$, with possibly $a = -\infty$ and/or $b = +\infty$.

(1) If $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} g(x) = 0$, then the limit $\lim_{x\to a^-} \frac{f(x)}{g(x)}$ is said to be an indeterminate form of the type $0/0$.
(2) If $\lim_{x\to a^-} f(x) = \pm\infty$ and $\lim_{x\to a^-} g(x) = \pm\infty$, then the limit $\lim_{x\to a^-} \frac{f(x)}{g(x)}$ is said to be an indeterminate form of the type ∞/∞.
(3) If $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} g(x) = +\infty$, then the limit $\lim_{x\to a^-} f(x) - g(x)$ is said to be an indeterminate form of the type $\infty - \infty$.
(4) If $\lim_{x\to a^-} f(x) = 0$ and $\lim_{x\to a^-} g(x) = \pm\infty$, then the limit $\lim_{x\to a^-} f(x)g(x)$ is said to be an indeterminate form of the type $0 \cdot \infty$.
(5) If $\lim_{x\to a^-} f(x) = 1$ and $\lim_{x\to a^-} g(x) = \infty$, then the limit $\lim_{x\to a^-} f(x)^{g(x)}$ is said to be an indeterminate form of the type 1^∞.
(6) If $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} = 0$, then the limit $\lim_{x\to a^-} f(x)^{g(x)}$ is said to be an indeterminate form of the type 0^0.

Similar definitions are made for $\lim_{x\to b^-}$.

The reason underlying the terminology is that, in each of the cases of Definition 3.1.1, nothing can be directly deduced, only from the assumptions on f and g, about the limit under consideration. Further investigation is needed, as illustrated in the upcoming example.

Example 3.1.2. In this example, we present functions f and g satisfying $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty$, where the following possibilities occur:

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 0 , \quad \lim_{x\to+\infty} \frac{f(x)}{g(x)} = +\infty , \quad \lim_{x\to+\infty} \frac{f(x)}{g(x)} = \sqrt{\pi} .$$

(1) Let $f(x) = x$, $g(x) = x^2$; then $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty$, and
$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{x}{x^2} = \lim_{x\to+\infty} \frac{1}{x} = 0 \,.$$
(2) Let $f(x) = x^2$, $g(x) = x$; then $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty$, and
$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{x^2}{x} = \lim_{x\to+\infty} x = +\infty \,.$$
(3) Let $f(x) = \sqrt{\pi}x$, $g(x) = x$; then $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty$, and
$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{\sqrt{\pi}x}{x} = \lim_{x\to+\infty} \sqrt{\pi} = \sqrt{\pi} \,.$$

Clearly, by appropriately modifying the last example, we can concoct two functions f, g satisfying $\lim_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} g(x) = +\infty$ and $\lim_{x\to+\infty} f(x)/g(x) = c$, where c is an given real number.

In the foregoing example, elementary algebraic simplifications allowed to eliminate the indeterminate form ∞/∞ arising each time, and thus evaluate the limit. A fundamental tool enabling us to deal with indeterminate forms of the type $0/0$ and ∞/∞ is what is known as *l'Hôpital's rule*, which permits to replace, for the purpose of computing limits, a quotient of two functions by the quotient of their derivatives.

Theorem 3.1.3 (l'Hôpital's rule). *Let $-\infty \le a < b \le +\infty$, $f, g\colon (a, b) \to \mathbb{R}$ differentiable functions. Assume that $g'(x) \neq 0$ for all $x \in (a, b)$ and*
$$\lim_{x\to a^-} \frac{f'(x)}{g'(x)} = L \,,$$
where $L \in \mathbb{R}$, $L = +\infty$ or $L = -\infty$. If

- *either* $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} g(x) = 0$ *(indeterminate form $0/0$)*
- *or* $\lim_{x\to a^-} g(x) = +\infty$ *or* $-\infty$ *(potential indeterminate form ∞/∞),*

then
$$\lim_{x\to a^-} \frac{f(x)}{g(x)} = L \,.$$
An analogous statements holds when considering $\lim_{x\to b^-} \frac{f'(x)}{g'(x)}$ *and* $\lim_{x\to b^-} \frac{f(x)}{g(x)}$.

Remark 3.1.4. In order to make sense of the limit
$$\lim_{x\to a^-} \frac{f(x)}{g(x)}$$
considered in Theorem 3.1.3, we should assume $g(x) \neq 0$ for $x \in (a, b)$, or at the very least for all x sufficiently close to a. However, there is no need to do so explicitly as this is implied by the other assumptions already in place for g:

- if $\lim_{x\to a^-} g(x) = +\infty$, then by definition of limit we certainly have $g(x) \ge 1$ for all $x \in (a, b)$ sufficiently close to a (and 1 can be replaced by any positive real number in such a statement), so that in particular $g(x) \neq 0$ for all such x. Similarly, if $\lim_{x\to a^-} g(x) = -\infty$, then $g(x) \le -1$ for all $x \in (a, b)$ sufficiently close to a, and thus $g(x) \neq 0$ for all such x.
- If $\lim_{x\to a^-} g(x) = 0$, then we make use of the second assumption $g'(x) \neq 0$ for all $x \in (a, b)$. Let's extend g by continuity at a by setting $g(a) = 0$, and fix a point $c \in (a, b)$. We regard the continuous function g on the closed interval $[a, c]$: g is continuous on it, and differentiable on (a, c) by assumption, with $g'(x) \neq 0$ for all $x \in (a, c)$. Now, if y is any point in $(a, c]$, the the mean value theorem (see Theorem B.3.1 in the Appendix) gives that there exists $\xi \in (a, y)$ such that
$$g(y) - g(a) = g'(\xi)(y - a) \,.$$

Now we know that $g(a) = 0$, $g'(\xi) \neq 0$ and $y - a \neq 0$, so that we deduce from the previous equality $g(y) = g(y) - g(a) \neq 0$.

We have thus shown that $g(y) \neq 0$ for all $y \in (a, c]$, so that it makes sense to consider the limit $\lim_{x\to a^-} f(x)/g(x)$.

EXAMPLE 3.1.5. Let's evaluate

$$\lim_{x\to+\infty} \frac{3x+5}{e^{8x}} \,.$$

Direct substitution leads to the indeterminate form ∞/∞. Observe also that $(e^{8x})' = 8e^{8x} \neq 0$ for every $x \in \mathbb{R}$. Also, we can compute

$$\lim_{x\to+\infty} \frac{(3x+5)'}{(e^{8x})'} = \lim_{x\to+\infty} \frac{3}{8e^{8x}} = 0 \,.$$

Hence we are under the assumptions of Theorem 3.1.3, which then yields

$$\lim_{x\to+\infty} \frac{3x+5}{e^{8x}} = \lim_{x\to+\infty} \frac{(3x+5)'}{(e^{8x})'} = 0 \,.$$

EXAMPLE 3.1.6. Let us now consider

$$\lim_{x\to+\infty} \frac{x^2+2x+1}{e^{x}} \,.$$

We are again in the case of an indeterminate form ∞/∞. Let's compute instead the limit of the quotient of the derivatives:

$$\lim_{x\to+\infty} \frac{(x^2+2x+1)'}{e^{x}} = \lim_{x\to+\infty} \frac{2x+2}{e^{x}} \,,$$

which again leads to an indeterminate form ∞/∞. However, we can take derivatives once more, and compute instead

$$\lim_{x\to+\infty} \frac{(2x+2)'}{(e^{x})'} = \lim_{x\to+\infty} \frac{2}{e^{x}} = 0 \,.$$

By Theorem 3.1.3, we deduce first that

$$\lim_{x\to+\infty} \frac{2x+2}{e^{x}} = \lim_{x\to+\infty} \frac{(2x+2)'}{(e^{x})'} = 0 \,,$$

and then, once again by Theorem 3.1.3, that

$$\lim_{x\to+\infty} \frac{x^2+2x+1}{e^{x}} = \lim_{x\to+\infty} \frac{(x^2+2x+1)'}{(e^{x})'} = 0 \,.$$

EXAMPLE 3.1.7. We wish to evaluate

$$\lim_{x\to\pi/2} \frac{\cos^2(x)}{1-\sin x} \,.$$

Direct substitution of $\cos^2(\pi/2) = 0$ and $\sin(\pi/2) = 1$ yields the indeterminate form $0/0$. Let's try to compute the limit of the quotient of the derivatives instead:

$$\lim_{x\to\pi/2} \frac{(\cos^2(x))'}{(1-\sin x)'} = \lim_{x\to\pi/2} \frac{-2\cos x \sin x}{-\cos x} = \lim_{x\to\pi/2} 2\sin x = 2 \,,$$

where the latter simplification in the fraction is possible as $\cos x \neq 0$ for all x close to (but not equal to[1]) $\pi/2$. Theorem 3.1.3 thus delivers

$$\lim_{x\to\pi/2} \frac{\cos^2(x)}{1-\sin x} = \lim_{x\to\pi/2} \frac{(\cos^2(x))'}{(1-\sin x)'} = 2 \,.$$

At times, it is possible to apply Theorem 3.1.3 when dealing with indeterminate forms other than $0/0$ or ∞/∞, reducing matters to one of the latter two via algebraic manipulations.

[1]Recall that the limit doesn't care about the specific value of the involved function at the given point.

EXAMPLE 3.1.8. We want to find

$$\lim_{x\to 0^+} x\log x\ .$$

Here we are faced with an indeterminate form $0\cdot\infty$. However, this is easily transformed into an indeterminate form of the type ∞/∞, amenable to an application of Theorem 3.1.3:

$$x\log x = \frac{\log x}{1/x}\ ,$$

where now both the numerator and the denominator tend to ∞ as $x\to 0^+$. Let's compute the limit of the quotient of the derivatives:

$$\lim_{x\to 0^+}\frac{(\log x)'}{(1/x)'} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0\ .$$

It follows that

$$\lim_{x\to 0^+} x\log x = 0\ .$$

EXAMPLE 3.1.9. Let's compute

$$\lim_{x\to+\infty} e^{3x}-x^3\ .$$

This is an indeterminate form of the type $\infty-\infty$. We may however factor the e^{3x} out, and get

$$\lim_{x\to+\infty} e^{3x}-x^3 = \lim_{x\to+\infty} e^{3x}\left(1-\frac{x^3}{e^{3x}}\right)\ . \tag{3.1.1}$$

Now, applying Theorem 3.1.3 three times to $\lim_{x\to+\infty} x^3/e^{3x}$, we obtain

$$\lim_{x\to+\infty}\frac{x^3}{e^{3x}} = \lim_{x\to+\infty}\frac{3x^2}{3e^{3x}} = \lim_{x\to\infty}\frac{2x}{3e^{3x}} = \lim_{x\to+\infty}\frac{2}{9e^{3x}} = 0\ .$$

Therefore, we conclude that $\lim_{x\to+\infty} 1-x^3/e^{3x} = 1$ and thus from (3.1.1) that

$$\lim_{x\to+\infty} e^{3x}-x^3 = +\infty;\ ,$$

since $\lim_{x\to+\infty} e^{3x} = +\infty$.

When dealing with indeterminate forms of the type 0^0 or 1^∞, it is often convenient to use the equality $a^b = e^{\log(a^b)} = e^{b\log a}$, for $a>0, b\in\mathbb{R}$, which often leads to an expression that can be treated with Theorem 3.1.3.

EXAMPLE 3.1.10. We wish to compute

$$\lim_{x\to 0^+} x^x\ .$$

We have an indeterminate form of the type 0^0. We write

$$x^x = e^{\log(x^x)} = e^{x\log x}\ .$$

Now we have already computed in Example 3.1.8 that $\lim_{x\to 0^+} x\log x = 0$; from this, we readily get

$$\lim_{x\to 0^+} x^x = 1\ .$$

EXAMPLE 3.1.11. We evaluate

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x\ .$$

The indeterminate form arising here is 1^∞. We rewrite the function as

$$\left(1+\frac{1}{x}\right)^x = e^{x\log\left(1+\frac{1}{x}\right)} = e^{\frac{\log\left(1+\frac{1}{x}\right)}{1/x}}\ , \tag{3.1.2}$$

where now $\lim_{x\to+\infty} \frac{\log\left(1+\frac{1}{x}\right)}{1/x}$ leads to an indeterminate form $0/0$. Applying Theorem 3.1.3, we have

$$\lim_{x\to+\infty} \frac{\log\left(1+\frac{1}{x}\right)}{1/x} = \lim_{x\to+\infty} \frac{\frac{1}{1+\frac{1}{x}}(-1/x^2)}{-1/x^2} = \lim_{x\to+\infty} \frac{1}{1+\frac{1}{x}} = 1\ ;$$

hence, from (3.1.2) we obtain

$$\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = e\ .$$

3.2. Growth rates of functions

In Example 3.1.9 we deal with two functions $f(x) = e^{3x}$, $g(x) = x^3$, both tending to $+\infty$ as $x \to +\infty$, whose difference $f(x) - g(x)$ also tends to $+\infty$ as $x \to +\infty$. This indicates that there is a hierarchy between different ways, or *speeds*, at which a given function can tend to ∞ at ∞. In this specific example, it is natural to deduce from the fact that $f(x) - g(x) \to +\infty$ as $x \to +\infty$ that $f(x)$ tends to $+\infty$ much faster than $g(x)$ does: by definition of the limit, for any real number $M > 0$, we the inequality $f(x) \geq g(x) + M$ holds for any sufficiently large x. More is true, namely

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{e^{3x}}{x^3} = +\infty\ , \tag{3.2.1}$$

as given by an application of Theorem 3.1.3 analogous to Examples 3.1.5 and 3.1.6. This quantifies the fact that e^{3x} is much larger than x^3 in an even stronger way with respect to the limit of the difference: indeed, (3.2.1) amounts to the fact that, for any real number $M > 0$, the inequality $f(x) \geq Mg(x)$ holds for any sufficiently large x.

It is convenient to set up a notion of comparison between the *growth rates* of two functions tending to ∞.

Definition 3.2.1 (Comparison of growth rates). Let f, g be two functions with

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty\ .$$

We say that f **grows faster than** g, or that g **grows more slowly than** f, and write $f \gg g$ or $g \ll f$, if

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = +\infty$$

or, equivalently, if

$$\lim_{x\to+\infty} \frac{g(x)}{f(x)} = 0\ .$$

Using Theorem 3.1.3 and elementary algebra, we can easily establish the folloiwng ranking of growth rates of commonly occcurring functions. Let $0 < r < s$ $1 < a < b$ be real numbers: then

$$\log x \ll x^r \ll x^s \ll a^x \ll b^x\ . \tag{3.2.2}$$

We explicate why the previous hold: it is immediate to see that

$$\lim_{x\to+\infty} \frac{x^s}{x^r} = \lim_{x\to+\infty} x^{s-r} = +\infty\ , \qquad \lim_{x\to+\infty} \frac{b^x}{a^x} = \lim_{x\to+\infty} \left(\frac{b}{a}\right)^x = +\infty\ ,$$

recalling that $s > r$ and $b > a$. Furthermore, Theorem 3.1.3 gives both

$$\lim_{x\to+\infty} \frac{x^r}{\log x} = \lim_{x\to+\infty} \frac{rx^{r-1}}{1/x} = \lim_{x\to+\infty} rx^r = +\infty$$

and

$$\lim_{x\to+\infty} \frac{a^x}{x^s} = \lim_{x\to+\infty} \frac{a^x \log a}{sx^{s-1}} = \cdots = \lim_{x\to+\infty} \frac{a^x(\log a)^{\lfloor s\rfloor}}{s(s-1)\cdots(s-\lfloor s\rfloor)x^{s-\lfloor s\rfloor-1}} = \lim_{x\to+\infty} \frac{xa^x(\log a)^{\lfloor s\rfloor}}{x^{s-\lfloor s\rfloor}} = +\infty$$

where $\lfloor s \rfloor$ denotes the largest integer smaller or equal to s, so that in particular $s - \lfloor s \rfloor < 1$.

Let us now examine a few properties of the relation $f(x) \gg g(x)$ just introduced, which point to the fact that, despite the similar-looking symbol, it doesn't enjoy the same properties as the standard order relation $a > b$ between two real numbers.

- If $f(x) \gg g(x)$ and $g(x) \gg h(x)$, then $f(x) \gg h(x)$; to see this, compute
$$\lim_{x\to+\infty} \frac{f(x)}{h(x)} = \lim_{x\to+\infty} \frac{f(x)}{g(x)}\frac{g(x)}{h(x)} = \lim_{x\to+\infty} \frac{f(x)}{g(x)} \lim_{x\to+\infty} \frac{g(x)}{h(x)} = +\infty \ .$$
- If $f(x) \gg g(x)$ and $c > 0$ is a real number, then $f(x) \gg cg(x)$ and $cf(x) \gg g(x)$; to see this, compute
$$\lim_{x\to+\infty} \frac{f(x)}{cg(x)} = \frac{1}{c} \lim_{x\to+\infty} \frac{f(x)}{g(x)} = +\infty \ ,$$
and similarly for the limit of $cf(x)/g(x)$. Thus, multiplication by constants does not alter the relative growth rate of two functions.
- If $f(x) \gg g(x)$ and $f(x) \gg h(x)$, then $f(x) \gg g(x) + h(x)$; to see this, observe that
$$\lim_{x\to+\infty} \frac{g(x)+h(x)}{f(x)} = \lim_{x\to+\infty} \frac{g(x)}{f(x)} + \lim_{x\to+\infty} \frac{h(x)}{f(x)} = 0 + 0 = 0 \ .$$
Thus, summing two functions growing more slowly than a third function f results in a new function which still grows more slowly than f.
- If $f(x) \gg h(x)$, $g(x) \gg l(x)$ and
$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L \ ,$$
where L is either a nonnegative real number or $L + \infty$, then
$$\lim_{x\to+\infty} \frac{f(x)+h(x)}{g(x)+l(x)} = L \ .$$
To see this, observe that we can write
$$\frac{f(x)+h(x)}{g(x)+l(x)} = \frac{f(x)(1+\frac{h(x)}{f(x)})}{g(x)(1+\frac{l(x)}{g(x)})} = \frac{f(x)}{g(x)} \frac{1+\frac{h(x)}{f(x)}}{1+\frac{l(x)}{g(x)}} \ ,$$
whence
$$\lim_{x\to+\infty} \frac{f(x)+h(x)}{g(x)+l(x)} = L \lim_{x\to+\infty} \frac{1+\frac{h(x)}{f(x)}}{1+\frac{l(x)}{g(x)}} = L \cdot 1 = L \ .$$
Therefore, when computing limits of quotients of sums of functions, only the two summands that grow the most rapidly, one for the numerator and one for the denominator, matter.

Let us see how arguing with growth rates can help evaluate limits rapidly.

EXAMPLE 3.2.2. Suppose we want to evaluate
$$\lim_{x\to+\infty} \frac{e^x + x^{100} + 200\log x}{3e^x + 2^x + 300x^2} \ .$$
We have $e^x \gg x^{100}$ and $e^x \gg 200\log x$, whence $e^x \gg x^{100} + 200\log x$. Similarly, $e^x \gg 2^x$ (as $e > 2$) and $e^x \gg 300x^2$, whence $3e^x \gg 2^x$ and $3e^x \gg 300x^2$ and so $3e^x \gg 2^x + 300x^2$. We conclude that
$$\lim_{x\to+\infty} \frac{e^x + x^{100} + 200\log x}{3e^x + 2^x + 300x^2} = \lim_{x\to+\infty} \frac{e^x}{3e^x} = \frac{1}{3} \ .$$

EXAMPLE 3.2.3. Let's compute
$$\lim_{x\to+\infty} \frac{2^{3x} + 3^{2x} + x}{e^{2x} + \log x} \ .$$

As $2^3 < 3^2$, we have $3^{2x} \gg 2^{3x}$; also, $3^{2x} \gg x$ and thus $3^{2x} \gg 2^{3x} + x$. Furthermore, $e^{2x} \gg \log x$, so that

$$\lim_{x \to +\infty} \frac{2^{3x} + 3^{2x} + x}{e^{2x} + \log x} = \lim_{x \to +\infty} \frac{3^{2x}}{e^{2x}} = +\infty \ ,$$

where the last step is a consequence of the fact that $3^2 > e^2$.

Remark 3.2.4. Care is needed when computing limits with growth rates. Suppose, for instance, we would like to compute

$$\lim_{x \to +\infty} \frac{\log (x^3)}{\log x} \ .$$

We do have $x^3 \gg x$, but this does not imply that $\log (x^3) \gg \log x$. However, a simple algebraic manipulation delivers

$$\lim_{x \to +\infty} \frac{\log (x^3)}{\log x} = \lim_{x \to +\infty} \frac{3 \log x}{\log x} = 3 \ .$$

CHAPTER 4

Elements of probability theory

4.1. Continuous random variables

Definition 4.1.1 (Continuous random variable). A random quantity X, taking real values, is a **continuous random variable** if there is a continuous non-negative function $p\colon \mathbb{R} \to \mathbb{R}$ with

$$\int_{-\infty}^{+\infty} p(x)\ \mathrm{d}x = 1$$

such that, for every $a < b \in \mathbb{R}$, the probability that X liesin the interval $[a, b]$ is given by

$$\mathbf{P}(a \leq \mathsf{X} \leq b) = \int_a^b p(x)\ \mathrm{d}x = 1\ . \tag{4.1.1}$$

The function p is called the **probability density function** associated to the random variable X.

Remark 4.1.2. Let us briefly comment about the assumptions imposed on the function p. The continuity requirement ensures that the definite integral $\int_a^b p(x)\ \mathrm{d}x$, which we want to define as the probability that X takes values in $[a, b]$, is well-defined for every $a < b \in \mathbb{R}$. The assumption that $p \geq 0$ is natural if we appeal to the intuitive notion of the probability of a random event occurring, which should be a real number between 0 and 1. If there was a point $x_0 \in \mathbb{R}$ with $p(x_0) < 0$, then by continuity $p(y) < 0$ for all y in a small interval $[a, b]$ containing x_0, so that we would have

$$\mathbf{P}(a \leq \mathsf{X} \leq b) = \int_a^b p(x)\ \mathrm{d}x < 0\ ,$$

which is definitely inconsistent with any heurisitic notion of probability.

Similarly, the requirement $\int_{-\infty}^{\infty} p(x)\ \mathrm{d}x$ corresponds to the intuitive plain fact that the random quantity X takes for sure some real value, in other words, $\mathbf{P}(-\infty < \mathsf{X} < +\infty) = 1$. Though we don't have the mathematical foundations of probability theory at our disposal in this course, it would be possible to deduce from the condition (4.1.1) that the equality

$$\mathbf{P}(a \leq \mathsf{X} \leq b) = \int_a^b p(x)\ \mathrm{d}x$$

extends to the case $a = -\infty$ or $b = +\infty$ (where both can happen simultaneously), interpreting the resulting integral as an improper integral (see Section **??**). Thus, we have coherently

$$\mathbf{P}(\mathsf{X} \in \mathbb{R}) = \mathbf{P}(-\infty < \mathsf{X} < +\infty) = \int_{-\infty}^{\infty} p(x)\ \mathrm{d}x = 1\ .$$

Definition 4.1.3 (Expected value of a random variable). Let X be a continuous random variable with probability density function $p(x)$. The **mean** or **expected value** of X is the quantity

$$\mu(\mathsf{X}) = \int_{-\infty}^{+\infty} x p(x)\ \mathrm{d}x\ .$$

Observe that $\mu(\mathsf{X})$ can be either finite or infinite, according to whether the improper integral defining it converges or diverges.

Example 4.1.4. Consider the function $p(x) = Ce^{-x/r}$ for $x \geq 0$, where $r > 0$ is a parameter. First, we would like to find the value of C making p into a probability density function on the given domain $[0, +\infty)$, so that we can associate to p a random variable X taking values in $[0, +\infty)$. First, we need to have $p(x) \geq 0$ for every $x \geq 0$, and since $e^{-x/r} > 0$ we must have $C \geq 0$. Secondly, we need

$$1 = \int_0^{\infty} p(x)\, \mathrm{d}x = C \int_0^{-\infty} e^{-x/r}\, \mathrm{d}x = C \lim_{t\to+\infty} \int_0^t e^{-x/r}\, \mathrm{d}x = C \lim_{t\to+\infty} -re^{-x/r}\Big|_{x=0}^{x=t}$$
$$= Cr \lim_{t\to+\infty} (1 - e^{-t/r}) = Cr\ .$$

Hence, we must have $C = 1/r$ in order for $p(x)$ to be a probability density function.

We now want to find the expected value of the random variablle X associated to the given probability density function $p(x) = \frac{1}{r}e^{-x/r}$, $x \geq 0$. We compute

$$\mu(\mathsf{X}) = \int_0^{+\infty} xp(x)\, \mathrm{d}x = \frac{1}{r} \lim_{t\to+\infty} \int_0^t xe^{-x/r}\, \mathrm{d}x = \frac{1}{r} \lim_{t\to+\infty} \left(-rxe^{-x/r}\Big|_{x=0}^{x=t} + r \int_0^t e^{-x/r}\, \mathrm{d}x \right)$$
$$= \frac{1}{r} \lim_{t\to+\infty} -rte^{-t/r} - r^2(e^{-t/r} - 1)\ ,$$

where we used integration by parts to compute $\int_0^t xe^{-x/r}\, \mathrm{d}r$. The last displayed limit is evaluated by means of Theorem 3.1.3: we have

$$\lim_{t\to+\infty} te^{-t/r} = \lim_{t\to+\infty} \frac{t}{e^{t/r}} = \lim_{t\to+\infty} \frac{1}{\frac{1}{r}e^{t/r}} = 0\ ,$$

so that

$$\mu(\mathsf{X}) = \frac{1}{r}\left(r^2 - \lim_{t\to+\infty} rte^{-t/r} - \lim_{t\to+\infty} r^2 e^{-t/r} \right) = r\ .$$

The function $p(x) = \frac{1}{r}e^{-x/r}$, $x \geq 0$ is called the **exponential probability density function with mean** r: it is adopted to model[1] random phenomena such that earthquakes, waiting times, failure times etc.

4.2. A fundamental example: the Gaussian distribution

The *Gaussian probability density function*, also known as the *Gaussian distribution* or *normal distribution*, is arguably the single most important example of probability density function of a continuous random variable.

Definition 4.2.1 (Gaussian distribution). Fix two parameters $\mu, \sigma \in \mathbb{R}$ with $\sigma > 0$. The **Gaussian distribution**, or **normal distribution**, with **mean** μ **and standard deviation** σ is the probability density function on $\mathbb{R}$ given as

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\ , \quad x \in \mathbb{R}. \tag{4.2.1}$$

The graph of the function $p(x)$ in (4.2.1) is the familiar *bell-shaped curve*, which governs the distribution of a wide variety of data. Examples are illustrated in Figures 4.1, 4.2, 4.3. The value of μ indicates the displacement, along the x-axis, of the top of the bell curve, whereas the value of σ points to the "flatness" of the bell curve: the higher σ is, the flatter the graph is.

[1] The exponential probability density function is a convenient way of modelling all those random phenomena with a "loss of memory" property, that is, when knowledge that a certain event has not occurred for a certain amount of time does not affect the probability of having to wait another given amount of time before it occurs. In mathematical terms, if X denotes the random time at which the event under consideration occurs, then the probability that $\mathsf{X} > s + t$ *knowing that* $\mathsf{X} > s$ is exactly the same as $\mathbf{P}(\mathsf{X} > t)$, for any real numbers $s, t \geq 0$. This explains why the exponential probability density function is commonly adopted to model occurrence of earthquakes, for instance.

In order to properly formalize the loss-of-memory property just mentioned, the notion of conditional probability is needed, which is outside the purview of this course.

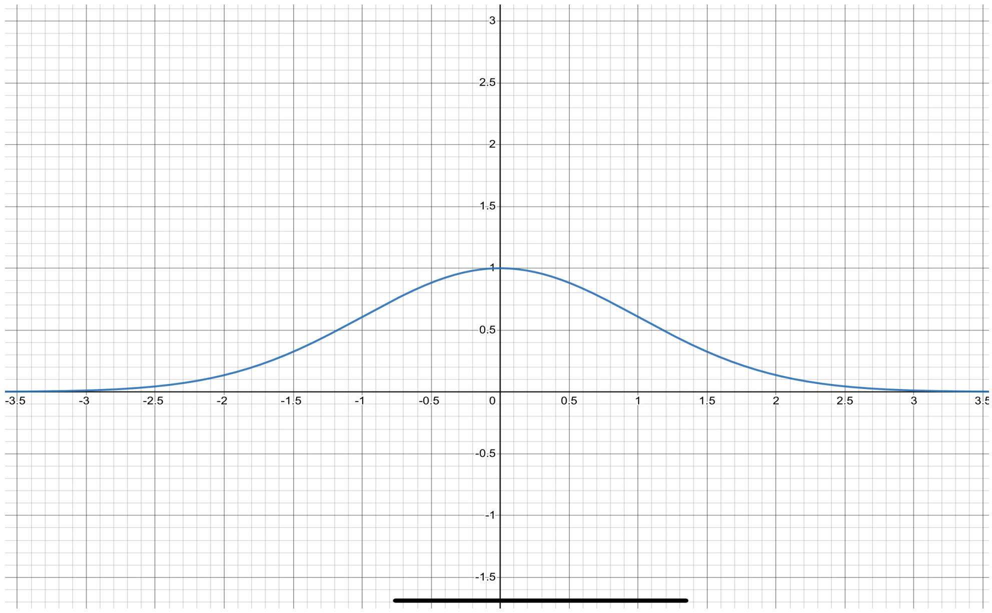

FIGURE 4.1. Graph of the Gaussian distribution for $\mu = 0$, $\sigma = 1$.

The value σ is called standard deviation as it measures how much the values of the Gaussian random variable X with probability density function p as in (4.2.1) deviate from the mean, the latter being given by the first parameter μ as we shall shortly demonstrate.

As $\sigma > 0$ and $e^t \geq 0$ for every $t \in \mathbb{R}$, we see that $p(x) \geq 0$ for every $x \in \mathbb{R}$, which, as seen in Section 4.1 is one of the conditions needed for a function p to be a probability density function. In fact, we have $p(x) > 0$ for every $x \in \mathbb{R}$, which by continuity of p entails that, if X is a random variable with probability density function p (called, accordingly, a **Gaussian random variable** or **normal random variable**),

$$\mathbf{P}(a \leq \mathsf{X} \leq b) = \int_a^b p(x)\ \mathrm{d}x > 0$$

for every $a < b \in \mathbb{R}$; this is to say that a Gaussian random variable has a strictly positive probability of taking values in any given non-degenerate interval of $\mathbb{R}$.

REMARK 4.2.2. For an *exponential random variable*, namely a random variable Y with associated probability density function $p(x) = \frac{1}{r}e^{-x/r}$, $x \geq 0$, examined in Example 4.1.4, the same property is true: for every $0 \leq a < b$, $\mathbf{P}(a \leq \mathsf{Y} \leq b) > 0$.

As far as the second characterizing property of a probability density function is concerned, namely the fact that $\int_{-\infty}^{+\infty} p(x)\ \mathrm{d}x = 1$, this is also verified by the function $p(x)$ in (4.2.1), though we don't have here the appropriate tools to show it[2]; as we already pointed out in Remark 1.1.10, it is not possible to express an anti-derivative of the function e^{-x^2} with a closed formula involving standard functional operations and elementary functions, which prevents us from computing

$$\int_{-\infty}^{+\infty} e^{-x^2}\ \mathrm{d}x = 2 \lim_{t \to +\infty} \int_0^t e^{-x^2}\ \mathrm{d}x$$

as seen in Section 1.5. *A fortiori*, we are not able to explicitly evaluate

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\ \mathrm{d}x$$

[2]They will be developed in subsequent courses, dealing with *multivariable calculus*.

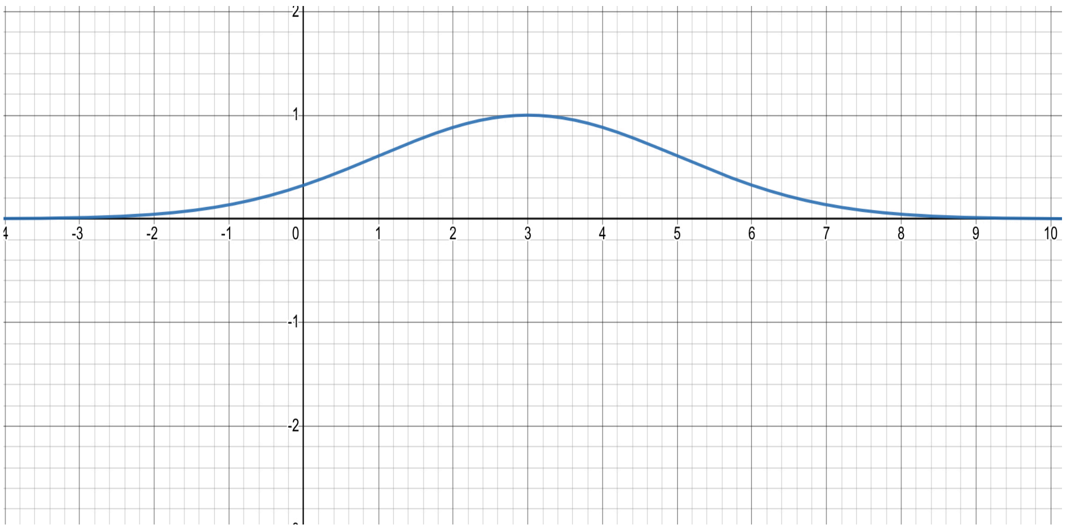

FIGURE 4.2. Graph of the Gaussian distribution for $\mu = 3$, $\sigma = 1$.

for any choice of parameters $\mu \in \mathbb{R}$, $\sigma > 0$, nor, for that matter, any definite integral of the form

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \,\mathrm{d}x \,,$$

which corresponds by definition to the probability that a Gaussian random variable of parameters μ and σ takes values in the interval $[a, b]$.

Let us see how the indefinite integrals

$$\int \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \,\mathrm{d}x \quad \text{and} \quad \int e^{-x^2} \,\mathrm{d}x \tag{4.2.2}$$

are closely related, so that not being able to explicitly evaluate the latter implies not being able to compute any of the former. By performing the substitution $u = (x - \mu)/\sigma$, $\mathrm{d}x = \sigma \mathrm{d}u$, we can readily transform

$$\int \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int e^{-u^2/2} \,\mathrm{d}u \,,$$

so that the integrals in (4.2.2) are connected by an elementary substitution.

Suppose now X is a **standard Gaussian random variable**, that is, a random variable with Gaussian distribution $p(x)$ associated to the parameters $\mu = 0$, $\sigma = 1$, hence given

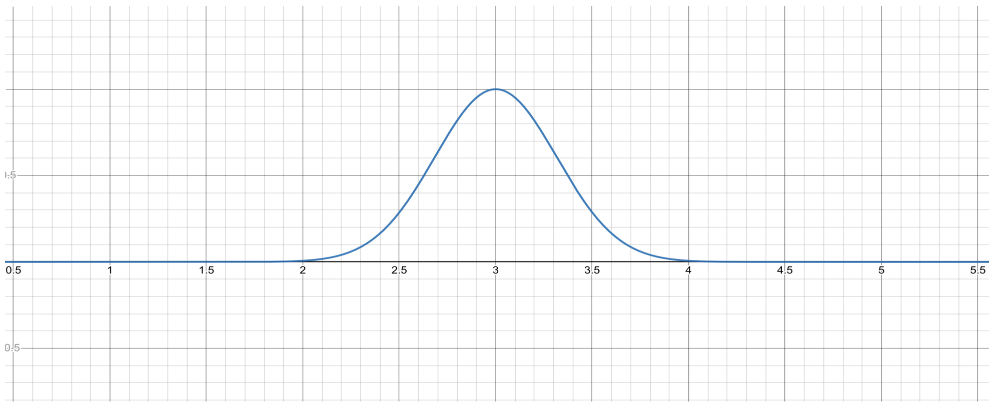

FIGURE 4.3. Graph of the Gaussian distribution for $\mu = 3$, $\sigma = 0.01$.

by $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}$. Let further Y be a Gaussian random variable with associated Gaussian distribution $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$, for arbitrary $\mu, \sigma \in \mathbb{R}$, $\sigma > 0$. For any $a < b \in \mathbb{R}$, we have, effecting the same substitution as above,

$$\mathbf{P}(a \leq \mathsf{Y} \leq b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\, \mathrm{d}x = \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, \mathrm{d}u = \mathbf{P}\Big(\frac{a-\mu}{\sigma} \leq \mathsf{X} \leq \frac{b-\mu}{\sigma}\Big) ;$$

this shows that understanding a standard Gaussian random variable allows, by a simple change of variables, to understand any Gaussian random variable.

Let us now turn to the evaluation of the mean of a Gaussian random variable X of parameters μ and σ. According to Definition 4.1.3, to compute the mean of X we need to evaluate

$$\begin{aligned} \mu(\mathsf{X}) &= \int_{-\infty}^{+\infty} x\, \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\, \mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{2\pi}} \lim_{t\to+\infty} \int_{\mu}^{\mu+t} x e^{-(x-\mu)^2/(2\sigma^2)}\, \mathrm{d}x + \int_{\mu-t}^{\mu} x e^{-(x-\mu)^2/(2\sigma^2)}\, \mathrm{d}x \end{aligned}$$

where in the second step we choose μ as the base point to exploit he symmetry of the graph of the function $p(x)$ with respect to the vertical axis $x = \mu$ in subsequent calculations. To evaluate the two integrals in the last displayed equation, we perform the substitution $u = \frac{x-\mu}{\sigma}$, leading to $x = \sigma u + \mu$, $\mathrm{d}x = \sigma \mathrm{d}u$, which yields

$$\int_{\mu}^{\mu+t} x e^{-(x-\mu)^2/(2\sigma^2)}\, \mathrm{d}x = \sigma\Big(\int_0^{t/\sigma} \sigma u e^{-u^2/2}\, \mathrm{d}u + \mu \int_0^{t/\sigma} e^{-u^2/2}\, \mathrm{d}u\Big)$$

and

$$\int_{\mu-t}^{\mu} x e^{-(x-\mu)^2/(2\sigma^2)}\, \mathrm{d}x = \sigma\Big(\int_{-t/\sigma}^{0} \sigma u e^{-u^2/2}\, \mathrm{d}u + \mu \int_{-t/\sigma}^{0} e^{-u^2/2}\, \mathrm{d}u\Big) .$$

Rearranging terms appropriately, we get

$$\mu(\mathsf{X}) = \frac{\sigma}{\sqrt{2\pi}} \lim_{t\to+\infty} \int_{-t/\sigma}^{t/\sigma} u e^{-u^2/2}\, \mathrm{d}u + \lim_{t\to+\infty} \mu \int_{-t/\sigma}^{t/\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, \mathrm{d}u . \tag{4.2.3}$$

Observe now that

$$\int_{-t/\sigma}^{t/\sigma} u e^{-u^2/2}\,\mathrm{d}u = 0 \tag{4.2.4}$$

for every $t > 0$, as it is the integral of an odd function over a symmetric interval around the origin. On the other hand,

$$\lim_{t\to+\infty} \int_{-t/\sigma}^{t/\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,\mathrm{d}u = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,\mathrm{d}u = 1 \tag{4.2.5}$$

by the property of a probability density function. Combining (4.2.3), (4.2.4) and (4.2.5) we deduce that

$$\mu(\mathsf{X}) = \mu\ ,$$

which justifies the terminology in Definition 4.2.1.

CHAPTER 5

Sequences and series

5.1. Sequences

Recall that $\mathbb{N} = \{0, 1, 2, \ldots, \ldots\}$ denotes the set of natural numbers.

DEFINITION 5.1.1 (Sequence). A **sequence**[1] is a function $a \colon \mathbb{N} \to \mathbb{R}$, or a function $a \colon \mathbb{N}_{\geq k} = \{k, k+1, \ldots, \ldots\} \to \mathbb{R}$ for a fixed integer $k \geq 1$.

We interpret a sequence as an ordered list of real numbers $\{a_0, a_1, \ldots, a_n, \ldots\}$ or $\{a_k, a_{k+1}, \ldots, a_{k+n}, \ldots\}$ ($k \geq 1$ an integer) by writing $a_0 = a(0)$, $a_1 = a(1), \ldots,\ a_n = a(n), \ldots$. A sequence is denote in compact form as $\{a_n\}_{n=0}^{\infty}$, or as $\{a_n\}_{n=k}^{\infty}$ if it starts from $k \geq 1$. The value a_n is called the **n-th term of the sequence**, and n is its **index**.

EXAMPLE 5.1.2. Given a function $f \colon [0, +\infty) \to \mathbb{R}$, setting $a_n = f(n)$ for every $n \in \mathbb{N}$ defines a sequence. In other words, restricting a real-valued function, defined on the set of non-negative real numbers, to the natural numbers defines a sequence in an obvious way. For instance, the following are examples of sequences obtained by restricting familiar real functions to the natural numbers:

$$a_n = n\ , \quad a_n = n^2\ , \quad a_n = \frac{1}{n}\ (n \geq 1), \quad a_n = e^n\ , \quad a_n = \sin(n\pi)\ , \quad a_n = \frac{\cos(n^3\pi)}{1-4n}\ .$$

EXAMPLE 5.1.3 (The factorial sequence). Here is the most important example of a sequence not arising from an elementary[2] real function: the sequence of *factorials*

$$a_n = n! = n(n-1)(n-2)\cdots 2 \cdot 1\ , \quad n \geq 1,$$

extended to $n = 0$ by convention[3] by setting $0! = 1$.

Henceforth, we shall consider sequences $\{a_n\}_{n=0}^{\infty}$, that is, starting with index 0, for notational convenience, but everything carries over unaffectedly to sequences $\{a_n\}_{n=k}^{\infty}$ for an arbitrary integer $k \geq 0$.

If $\{a_n\}_{n=0}^{\infty}$ is a sequence, we are primarily interested in its *behaviour at infinity*, namely in the behaviour of the values (terms) a_n for very large n. We are lead to the following natural adaptation of the concept of limit at infinity of a real function.

DEFINITION 5.1.4 (Convergence and divergence of a sequence). A sequence $\{a_n\}_{n=0}^{\infty}$ **converges** to a real number L if the following is satisfied: for any real number $\varepsilon > 0$ there exists an integer $M \geq 1$ such that $|a_n - L| \leq \varepsilon$ for every $n \geq M$. If $\{a_n\}_{n=0}^{\infty}$ does not converge to any real number, we say it **diverges**.

When $\{a_n\}_{n=0}^{\infty}$ converges to L, we write

$$\lim_{n\to\infty} a_n = L\ .$$

[1] The same notion can be generalized to functions defined on the natural numbers with values in an arbitrary set X; here we confine ourselves to real-valued sequences.

[2] It turns out that the factorial $n!$ can be extended to a continuous function $[0, +\infty) \to \mathbb{R}$, called the *Gamma function*; we shall never be concerned with such an extension.

[3] There are several reaons, chiefly stemming from extending naturally the validity of certain formulas involving factorials to the case $n = 0$, to adopt this convention.

EXAMPLE 5.1.5. The sequence $a_n = \frac{1}{n}$ converges to 0. To see this via Definition 5.1.4, let us fix an arbitrary real number $\varepsilon > 0$. We need to find an integer $M \geq 1$ such that $\frac{1}{n} \leq \varepsilon$ for every $n \geq M$. Now the latter inequality is satisfied precisely when $n \geq \frac{1}{\varepsilon}$, so that choosing any integer $M \geq \frac{1}{\varepsilon}$ does the job[4].

EXAMPLE 5.1.6. The sequence $a_n = \frac{1}{2^n}$ converges to 0. To see this, let us appeal once again to Definition 5.1.4, and fix an arbitrary real number $\varepsilon > 0$. We need to find an integer $M \geq 1$ such that $\frac{1}{2^n} \leq \varepsilon$ for every $n \geq M$. The latter inequality is satisfied precisely when $2^n \geq \frac{1}{\varepsilon}$, that is, when $n \geq \log_2(1/\varepsilon)$, so that it suffices to pick any integer $M \geq \log_2(1/\varepsilon)$.

Observe that there are two possible divergence cases.

(1) It may happen that $\{a_n\}_{n=0}^{\infty}$ **diverges to infinity**, when $\lim_{n\to\infty} a_n$ exists but is equal to $+\infty$ or to $-\infty$. Formally, the former case happens if, for every real number L, there is an integer $M \geq 1$ such that $a_n \geq L$ for every $n \geq M$, whereas the latter case happens if, for every real number L, there is an integer $M \geq 1$ such that $a_n \leq L$ for every $n \geq M$. For instance,

$$\lim_{n\to\infty} n = +\infty , \quad \lim_{n\to\infty} -n^3 + n^2 + n + 1 = -\infty .$$

(2) It may be the case that $\{a_n\}_{n=0}^{\infty}$ neither converges nor diverges to $\pm\infty$, that is, the limit does not exists: this occurs, for instance, when$a_n = (-1)^n$, since we then have

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases} ,$$

so that a_n does not approach any finite or infinite value as n tends to infinity.

In the following proposition, we list a few elementary properties of convergent sequences.

PROPOSITION 5.1.7. *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ be sequences.*

(1) If $\{a_n\}_{n=0}^{\infty}$ converges to L and c is a real number, then the sequence $\{ca_n\}_{n=0}^{\infty}$ converges to cL.

(2) If $\{a_n\}_{n=0}^{\infty}$ converges to L_1 and $\{b_n\}_{n=0}^{\infty}$ converges to L_2, then the sequence $\{a_n+b_n\}_{n=0}^{\infty}$ converges to $L_1 + L_2$.

(3) If $\{a_n\}_{n=0}^{\infty}$ converges to L_1 and $\{b_n\}_{n=0}^{\infty}$ converges to L_2, then the sequence $\{a_n b_n\}_{n=0}^{\infty}$ converges to $L_1 L_2$.

(4) If $a_n \leq b_n \leq c_n$ for every $n \geq 0$ and both $\{a_n\}_{n=0}$ and $\{c_n\}_{n=0}^{\infty}$ converge to the same limit L, then $\{b_n\}_{n=0}^{\infty}$ converges to L as well.

(5) If there is a function $f\colon [0, +\infty) \to \mathbb{R}$ such that $a_n = f(n)$ for all $n \geq 0$, and if

$$\lim_{x\to+\infty} f(x) = L$$

for some $L \in \mathbb{R}$, then $\{a_n\}_{n=0}^{\infty}$ converges to L.

EXAMPLE 5.1.8. In Examples 5.1.5 and 5.1.6 we saw that the sequences $1/n$ and $1/2^n$ converge to 0 by applying directly Definition 5.1.4. It is also possible to deduce this fact from the last assertion of Proposition 5.1.7, as we already know that the functions $f(x) = 1/x$ and $g(x) = 1/2^x$ tend to 0 as $x \to +\infty$.

EXAMPLE 5.1.9. The sequence $a_n = n^{1/n}$ converges to 1. Indeed, we have that

$$\lim_{x\to+\infty} x^{1/x} = \lim_{x\to+\infty} e^{\frac{1}{x}\log x} = e^0 = 1 ,$$

where the second-to-last step follows from (3.2.2).

[4]The fact that, given any real number x, there is an integer M which is larger than x is called the *archimedean property* of the natural numbers.

5.2. Bounded and monotonic sequences

We shall here introduce two important notions in the study of the convergence/divergence properties of sequences.

DEFINITION 5.2.1 (Bounded sequence). A sequence $\{a_n\}_{n=0}^{\infty}$ is

- **bounded from above** if there is a real number L such that $a_n \leq L$ for all $n \geq 0$,
- **bounded from below** if there is a real number L such that $a_n \geq L$ for all $n \geq 0$,
- and **bounded** if it is both bounded from above and bounded from below.

EXAMPLE 5.2.2. The sequence $a_n = 1 - n$ is bounded from above: indeed, $1 - n \leq 1$ for all $n \geq 0$. However, it is not bounded from below, since for every $L \in \mathbb{R}$ there is n such that $1 - n < L$, that is, such that $a_n < L$.

The sequence $a_n = n$ is bounded from below: indeed, trivially $n \geq 0$ for all $n \geq 0$. However, it is not bounded from above, as for every $L \in \mathbb{R}$ there is $n \geq 0$ such that $n > L$, that is, such that $a_n > L$.

The sequence $a_n = \sin(n^4\pi)$ is bounded, as $-1 \leq a_n \leq 1$ for all $n \geq 0$.

DEFINITION 5.2.3 (Monotonic sequence). A sequence $\{a_n\}_{n=0}^{\infty}$ is

- **increasing** if $a_n < a_{n+1}$ for all $n \geq 0$,
- **decreasing** if $a_n > a_{n+1}$ for all $n \geq 0$,
- and **monotonic** if it is either increasing or decreasing.

EXAMPLE 5.2.4. The sequence $a_n = n$ is increasing, since $n < n + 1$ for all $n \geq 0$.

The sequence $a_n = 3^{-n}$ is decreasing, since $2^{-(n+1)} < 2^{-n}$ for all $n \geq 0$.

The sequence $a_n = (-1)^n$ is not monotonic, since $a_n < a_{n+1}$ for every odd n and $a_n > a_{n+1}$ for every even n.

In the remainder of this section, we will investigate the relationship between the notions of convergene, boundedness and monotonicity for sequences.

THEOREM 5.2.5. *Every convergent sequence is bounded.*

PROOF. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence converging to a real number L. From Definition 5.1.4, we see in particular that, setting $\varepsilon = 1$, there is an integer $M \geq 1$ such that $|a_n - L| \leq 1$ for all $n \geq M$, that is,

$$L - 1 \leq a_n \leq L + 1 \quad \text{for all } n \geq M.$$

Set now

$$a_{\min} = \min\{a_0, a_1, \ldots, a_{M-1}\}\ , \quad a_{\max} = \max\{a_0, a_1, \ldots, a_{M-1}\}\ .$$

Then we have

$$\min\{a_{\min}, L - 1\} \leq a_n \leq \max\{a_{\max}, L + 1\} \quad \text{for all } n \geq 0,$$

which shows that $\{a_n\}_{n=0}^{\infty}$ is bounded from above and below, and thus bounded by definition. □

The converse of the previous theorem fails to hold, namely, it is not true that every bounded sequence is convergent.

EXAMPLE 5.2.6. The sequence $a_n = (-1)^n$ is bounded since $-1 \leq a_n \leq 1$ for all $n \geq 0$, but doesn't have any limit as n tends to infinity as it keeps oscillating between the values 1 and -1.

However, adding a further monotonicity assumption ensures convergence.

THEOREM 5.2.7. *Every bounded monotonic sequence is convergent.*

Proof. Let $\{a_n\}_{n=0}^{\infty}$ be a bounded monotonic sequence. Let us assume that it is increasing: the argument carries over easily to the decreasing case with the appropriate modifications.

We define the following quantity:

$$L = \min\{A \in \mathbb{R} : a_n \leq A \text{ for every } n \geq 0\} .$$

We claim that $\{a_n\}_{n=0}^{\infty}$ converges to L. First, observe that the set in (5.2) is non-empty, since by assumption $\{a_n\}_{n=0}^{\infty}$ is bounded from above and thus there exists $A \in \mathbb{R}$ such that $a_n \leq A$ for every $n \geq 0$. Clearly, any real number $B \geq A$ satisfies the same property. However, the fact that there is a minimal element L satisfying this property is far from obvious, and is actually a consequence of a fundamental property of the set of real numbers, which we will not develop formally in this course.

We now set out to prove the claimed convergence of the sequence to L, using Definition 5.1.4. To this end, we fix an arbitrary real number $\varepsilon > 0$; we need to find an integer $M \geq 1$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq M$. We first observe that there is an integer $M \geq 1$ such that $a_M \geq L - \varepsilon$; else, we would have $a_n \leq L - \varepsilon$ for all $n \geq 0$, which would contradictict the definition of L by yielding $L \leq L - \varepsilon$. Since $\{a_n\}_{n=0}^{\infty}$ is increasing, we further deduce that $L - \varepsilon \leq a_M < a_n$ for all $n \geq M$. On the other hand, the definition of L tells us in particular that $a_n \leq L$ for all $n \geq 0$. Combining the latter two facts, we get

$$L - \varepsilon \leq a_n \leq L \quad \text{for all } n \geq M,$$

which implies that $|a_n - L| \leq \varepsilon$ for all $n \geq M$, as desired. □

Example 5.2.8. A convergent sequence is not necessarily monotonic: for instance, the sequence $a_n = \frac{(-1)^n}{n}$ converges to 0 by the fourth property in Proposition 5.1.7, since $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$ for all $n \geq 0$ and both $-\frac{1}{n}$ and $\frac{1}{n}$ converge to 0. However, a_n keeps alternating between positive and negative values, and thus is not monotonic.

Example 5.2.9. There is no general relationship between the notions of boundedness and of monotonicity, which is to say that a sequence can be bounded and not monotonic, as the sequence $a_n = (-1)^n/n$ from the previous example, and monotonic but not bounded, as the sequence $a_n = n$.

5.3. Series

Intimately tied to the concept of sequence is the notion of *series*, to which we now turn. Series are particular instances of sequences, obtained by summing all the first n terms of a given sequence.

Definition 5.3.1 (Series). Given a sequence $\{a_n\}_{n=0}^{\infty}$, the **series** of terms $a_0, a_1, \ldots, a_n$ is the sequence $\{S_N\}_{N=0}^{\infty}$ defined as

$$S_0 = a_0 , \quad S_1 = a_0 + a_1 , \quad S_2 = a_0 + a_1 + a_2 , \quad \ldots , \quad S_N = \sum_{n=0}^{N} a_n , \quad \ldots .$$

The sequence $\{S_N\}_{N=0}^{\infty}$ is also called the **sequence of partial sums** of the given series. We denote the series $\{S_N\}_{N=0}^{\infty}$ by $\sum_{n=0}^{\infty} a_n$, and say that it **converges** if it does so as a sequence, and **diverges** otherwise. If a series converges towards a real number L, we write $\sum_{n=0}^{\infty} a_n = L$; if it diverges to $\pm\infty$, we write $\sum_{n=0}^{\infty} a_n = \pm\infty$.

Example 5.3.2. The series $\sum_{n=0}^{\infty} n$ has general term $a_n = n$. The sequence of partial sums is

$$S_N = \sum_{n=0}^{N} n = 0 + 1 + 2 + 3 + \cdots + N , \quad N \in \mathbb{N}.$$

Observe, in particular, that $S_N \geq N$ for every $N \geq 0$, and since the sequence $\{N\}_{N=0}^{\infty}$ diverges to $+\infty$, so does $\{S_N\}_{N=0}^{\infty}$; therefore, by definition, the series $\sum_{n=0}^{\infty} n$ is divergent to infinity[5].

EXAMPLE 5.3.3. The series $\sum_{n=0}^{\infty} 1$ has general term $a_n = 1$. The sequence of partial sums is

$$S_N = \sum_{n=0}^{N} 1 = N + 1\,; \quad N \in \mathbb{N},$$

whence the series diverges to infinity, as $N + 1 \to +\infty$ as $N \to +\infty$.

EXAMPLE 5.3.4. The series $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ has general term $a_n = \frac{1}{n} - \frac{1}{n+1}$. The sequence of partial sums is

$$S_N = \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N} - \frac{1}{N+1}\,, \quad N \geq 1.$$

Note that all terms in the previous sum cancel out, except for the first and the last one, so that $S_N = 1 - \frac{1}{N+1}$ for all $N \geq 1$. Since $\lim\limits_{N\to\infty} 1 - \frac{1}{N+1} = 1$, we have that the series $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ converges, with $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1$.

EXAMPLE 5.3.5. The series $\sum_{n=0}^{\infty} (-1)^n$ has general term $a_n = (-1)^n$. The sequence of partial sums is

$$S_N = \sum_{n=0}^{N} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots + (-1)^N\,, \quad N \in \mathbb{N}.$$

Notice that $S_N = 0$ for all even $N \geq 0$, and $S_N = 1$ for all odd $N \geq 0$. Therefore, the sequence $\{S_N\}_{N=0}^{\infty}$ keeps alternating between the values 0 and 1, and thus does not admit any limit as $N \to \infty$. It follows by definition that the series $\sum_{n=0}^{\infty} (-1)^n$ diverges.

It is typically hard to find a closed, manageable formula for the N-th partial sum $S_N = \sum_{n=0}^{N}$ of a given series; in the absence of such a formula, we need to develop *ad hoc* methods, or *tests*, to study the convergence behaviour of series; as we are going to see, they can be put to good use in order to determine whether a series converges or diverges, however they tell us nothing, in the convergent case, about the precise limit of the series, which is only computable explicitly in very particular cases.

We conclude this section by rephrasing some of the content of Proposition 5.1.7 in the context of series.

PROPOSITION 5.3.6. *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two series.*

(1) If $\sum_{n=0}^{\infty} a_n = L$ and c is a real number, then $\sum_{n=0}^{\infty} c a_n = cL$.
(2) If $\sum_{n=0}^{\infty} a_n = L_1$ and $\sum_{n=0}^{\infty} b_n = L_2$, then $\sum_{n=0}^{\infty} a_n + b_n = L_1 + L_2$.
(3) Suppose $a_n = b_n$ for all but finitely many integers $n \geq 0$. Then the series $\sum_{n=0} a_n$ converges if and only if the series $\sum_{n=0}^{\infty}$ converges.
(4) If $a_n \geq 0$ for all $n \geq 0$, then either $\sum_{n=0}^{\infty} a_n$ converges or it diverges to infinity. The former case happens if and only if the sequence $\{S_N\}_{N=0}^{\infty}$ of partial sums $S_N = \sum_{n=0}^{N}$ is bounded.

REMARK 5.3.7. The third assertion formalizes the fact that *altering finitely many terms of a series does not change its convergence properties*: if we start with a series $\sum_{n=0}^{\infty} a_n$ converging to a real number L, and we replace finitely many of the a_n's with new real numbers a'_n, while leaving all the others unchanged, then the new series $\sum_{n=0}^{\infty} a_n$ will still converge, though possibly

[5] There is an explicit closed formula for the partial sums of such a series, given by $\sum_{n=0}^{N} n = \frac{N(N+1)}{2}$, from which the divergent behaviour of the series emerges even more directly. However, we were able to determine divergence without knowing such an explicit formula, which we will not prove in this course.

towards a different value. Similarly, if the original series $\sum_{n=0}^{\infty} a_n$ diverges, then so does the new series $\sum_{n=0}^{\infty} a'_n$.

Incidentally, observe that the same assertion is true for sequences: if $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences which differ only for finitely many terms, namely there is some integer $M > 0$ such that $a_n = b_n$ for all $n \geq M$, then the first sequence converges if and only if the second sequence converges, and in this case the limits are the same. This is a direct consequence of the definition of convergence of a sequence.

Proof. The first two assertions follow at once from the respective statements for sequences in Proposition 5.1.7, applied to the sequences of partial sums of the series under consideration.

Let us show the third assertion. By assumption, there is some integer $M \geq 1$ such that $a_n = b_n$ for all $n \geq M$. For all integers $N > M$, we split the partial sums $S_n = \sum_{n=0}^{N} a_n$, $S'_N = \sum_{n=0}^{N} b_n$ of the two series in the following way:

$$S_N = \sum_{n=0}^{M} a_n + \sum_{n=M+1}^{N} a_n \ , \quad S'_N = \sum_{n=0}^{M} b_n + \sum_{n=M+1}^{N} b_n \ ,$$

and call

$$S''_N = \sum_{n=M+1}^{N} a_n = \sum_{n=M+1}^{N} b_n \ .$$

If $\sum_{n=0}^{\infty} a_n$ converges, then by definition this means that there exists some real number L such that

$$L = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{M} a_n + S''_N = \sum_{n=0}^{M} a_n + \lim_{N\to\infty} S''_N \ ;$$

therefore,

$$\lim_{N\to\infty} S''_N = L - \sum_{n=0}^{M} a_n.$$

As a result,

$$\lim_{N\to\infty} S'_N = \lim_{N\to\infty} \sum_{n=0}^{M} b_n + S''_N = \sum_{n=0}^{M} b_n + \lim_{N\to\infty} S''_N = \sum_{n=0}^{M} b_n - \sum_{n=0}^{M} a_n + L \ ,$$

so that by definition the series $\sum_{n=0}^{\infty} b_n$ converges. Interchanging the roles of $\{a_n\}_{n=0}\infty$ and $\{b_n\}_{n=0}^{\infty}$ in the previous argument, we deduce the converse statement: if $\sum_{n=0}^{\infty} b_n$ converges, then so does $\sum_{n=0}^{\infty} a_n$.

We finally show the last assertion. Suppose $a_n \geq 0$ for all $n \geq 0$. Then the sequence of partial sums $S_N = \sum_{n=0}^{N} a_n$ is *non-decreasing*, namely satisfies $S_N \leq S_{N+1}$ for all $N \geq 0$. If $\{S_N\}_{N=0}^{\infty}$ is bounded, then it converges by Theorem 5.2.7 and[6] thus, by definition, the series $\sum_{n=0}^{\infty} a_n$ converges. Suppose instead that the sequence $\{S_N\}_{N=0}^{\infty}$ is not bounded; then it cannot converge by virtue of Theorem 5.2.5. The proof is concluded. □

5.4. Geometric series

We introduce a fundamental family of series in the following definition.

Definition 5.4.1 (Geometric series). A series $\sum_{n=0}^{\infty} a_n$ is called **geometric** of parameter r, where r is a real number, if $a_n = r^n$ for all $n \geq 0$.

Thus geometric series are those whose terms are obtained by successive integral powers of a given real number.

[6]Strictly speaking, Theorem 5.2.7 is phrased for monotonic sequenes, namely those satisfying either $S_n < S_{n+1}$ for all n or $S_n > S_{n+1}$ for all n; however, it is easy to realise that the argument carries over unaffectedly to the more general case of sequences satisfying either $S_n \leq S_{n+1}$ or $S_n \geq S_{n+1}$.

REMARK 5.4.2. Sometimes the name *geometric series* refers, more generally, to series of the form $\sum_{n=0}^{\infty} cr^n$, where c and r are real numbers.

What is particularly noteworthy about geometric series is that they represent one of the very few examples of series whose limit can be explicitly computed, via a simple closed formula for the sequence of partial sums.

THEOREM 5.4.3. *Let r be a real number. For every integer $N \geq 0$, it holds*

$$\sum_{n=0}^{N} r^n = \begin{cases} \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ N & \text{if } r = 1 \end{cases} .$$

As a consequence, the geometric series $\sum_{n=0}^{\infty} r^n$

- *converges to $\frac{1}{1-r}$ if $|r| < 1$ and*
- *diverges if $|r| \geq 1$.*

PROOF. The formula for the partial sums in the case $r = 1$ is immediate. The fact that $\sum_{n=0}^{\infty} r^n$ for $r = 1$ diverges follows directly.

Let us now assume that $r \neq 1$. We multiply the partial sum $\sum_{n=0}^{N} r^n$ by the quantity $1 - r$, so that the resulting sum contains a lot of terms which cancel out:

$$(1-r)\sum_{n=0}^{N} r_n = (1-r)(1+r+\cdots+r^N) = 1+r+\cdots+r^N - r - r^2 - \cdots - r^N - r^{N+1} = 1 - r^{N+1} ,$$

from which the claimed formula follows by dividing the first and the last term of the previous chain of equalities by the non-zero quantity $1 - r$.

Suppose now that $|r| < 1$. Then $\lim_{N\to\infty} r^{N+1} = 0$, and thus

$$\lim_{N\to\infty} \sum_{n=0}^{N} r^n = \lim_{N\to\infty} \frac{1 - r^{N+1}}{1 - r} = \frac{1}{1 - r} .$$

On the other hand, if $|r| \geq 1$ with $r \neq 1$, then the sequence $\{r^N\}_{N=0}^{\infty}$ does not converge, and thus the series $\sum_{n=0}^{\infty} r^n$ cannot converge. The same holds for $r = 1$, as in this case $\sum_{n=0}^{N} r^n = N + 1$, which tends to infinity as N does. □

EXAMPLE 5.4.4. We have

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - 1/2} = 2 .$$

From this we may deduce that $\sum_{n=1}^{\infty} 2^{-n} = \sum_{n=0}^{\infty} 2^{-n} - -1 = 1$; it is instructive to think about such equality in geometric terms: on the real line, start from the origin 0 and begin by adding a half of the segment $[0, 1]$, so as to reach $1/2$. Then add one half of the remaining segment $[1/2, 1]$, so as to reach $1/2 + 1/4$, and continue in this way. It is clear that, while after finitely many steps there will always be some remaining portion of the interval $[0, 1]$ which is not covered, *in the limit* as the number of steps tends to infinity every point in $[0, 1]$ gets covered, or in other words the infinite sum of all the lengths of the intervals added each time is precisely the length of the whole interval $[0, 1]$.

EXAMPLE 5.4.5. We compute

$$\sum_{n=0}^{\infty} \frac{2^n + 4 \cdot 5^n}{3^{2n}} .$$

We have

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{2n}} = \sum_{n=0}^{\infty} \left(\frac{2}{3^2}\right)^n = \frac{1}{1 - \frac{2}{9}} = \frac{9}{7}$$

and

$$\sum_{n=0}^{\infty} \frac{5^n}{3^{2n}} = \sum_{n=0}^{\infty} \left(\frac{5}{3^2}\right)^n = \frac{1}{1-\frac{5}{9}} = \frac{9}{4} \,,$$

whence by the first two properties in Proposition 5.3.6 we deduce that

$$\sum_{n=0}^{\infty} \frac{2^n + 4 \cdot 5^n}{3^{2n}} = \sum_{n=0}^{\infty} \frac{2^n}{3^{2n}} + 4 \sum_{n=0}^{\infty} \frac{5^n}{3^{2n}} = \frac{9}{7} + 4 \cdot \frac{9}{5} = \frac{297}{35} \,.$$

REMARK 5.4.6 (Geometric series with arbitrary starting point). A series of the form $\sum_{n=N}^{\infty} r^n$, where r is a real number and $N \geq 0$ is an integer, is also called a geometric series. The starting index is not 0 any longer, but rather N, explicitly

$$\sum_{n=N}^{\infty} r^n = r^N + r^{N+1} + r^{N+2} + \cdots + r^{N+n} + \cdots \,.$$

It is straightforward to deduce a formula for the infinite sum of such generalized geometric series, when $|r| < 1$, starting from the two formulas

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1-r} \,, \qquad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \,.$$

We proceed in two different ways, which plainly will lead to the same result.

First, we may compute $\sum_{n=N} r^n$ by first summing the whole series $\sum_{n=0}^{\infty} r^n$ and then subtracting the sum of the first N terms, that is,

$$\sum_{n=N}^{\infty} r^n = \sum_{n=0}^{\infty} r^n - \sum_{n=0}^{N-1} r^n = \frac{1}{1-r} - \frac{1-r^N}{1-r} = \frac{r^N}{1-r} \,.$$

Alternatively, we may observe that, for every integer $n \geq N$, we can split the power r^n as $r^N r^{n-N}$, where the first factor r^N does not depend on the index n any longer, and thus can be factored out of the summation, whereas the exponent $n - N$ in the second factor now runs from 0 to ∞; to be precise,

$$\sum_{n=N}^{\infty} r^n = \sum_{n=N}^{\infty} r^N r^{n-N} = r^N \sum_{n=N}^{\infty} r^{n-N} = r^N \sum_{m=0}^{\infty} r^m = \frac{r^N}{1-r} \,,$$

where in the second-to-last step we set $m = n - N$.

5.5. Convergence tests

5.5.1. Divergence test. The first method we shall be concerned with in the investigation of series is called the *divergence test*: as the name suggests, it gives a sufficient, easy-to-check condition for a series to diverge. However, the condition is by no means necessary; in other words, if it fails, then nothing can be inferred about the behaviour of the series, as will be shortly manifest by means of examples. We say that in such cases the test is *inconclusive*.

THEOREM 5.5.1 (Divergence test). *If the series $\sum_{n=0}^{\infty} a_n$ is convergent, then*

$$\lim_{n\to\infty} a_n = 0 \,.$$

Before proceeding with the proof, let us phrase the contrapositive of the theorem's statement explicitly: *any series $\sum_{n=0}^{\infty} a_n$ such that $\{a_n\}_{n=0}^{\infty}$ is either divergent or convergent to a non-zero limit must necessarily diverge.*

REMARK 5.5.2. Observe the subtlety residing in the fact that, in order to produce information about the behaviour of the series $\sum_{n=0}^{\infty} a_n$, which by definition is the behaviour of the sequence $\{S_N\}_{N=0}^{\infty}$ of partial sums $S_N = \sum_{n=0}^{N} a_n$, the divergence test resorts to the limiting behaviour of the original sequence $\{a_n\}_{n=0}^{\infty}$.

PROOF OF THEOREM 5.5.1. Suppose that $\sum_{n=0}^{\infty} a_n$ converges towards a real number L. We need to show that, for every real number $\varepsilon > 0$, there is some integer $M \geq 1$ such that

$$|a_n| \leq \varepsilon \quad \text{for all } n \geq M.$$

We apply the definition of convergence for series to the value $\varepsilon/2$: that is, we let $M \geq 1$ be an integer with the property that, for every $n \geq M$,

$$|S_n - L| = \left| \sum_{i=0}^{n} a_i - L \right| \leq \frac{\varepsilon}{2}.$$

The crucial, elementary observation at this point is that the n-th term a_n of the series can be obtained from the partial sums by taking the difference[7] $S_n - S_{n-1}$. Invoking the triangle inequality (see Theorem A.1.1 in the Appendix), we deduce that, whenever $n \geq M + 1$,

$$|a_n| = |S_n - S_{n-1}| = |S_n - L - (S_{n-1} - L)| \leq |S_n - L| + |S_{n-1} - L| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is what we wanted to show. □

Let us see how to apply the divergence test in practice.

EXAMPLE 5.5.3. We have already seen in Example 5.3.3 that the series $\sum_{n=0}^{\infty} 1$ diverges. This can also be inferred from Theorem 5.5.1: the general term of the series is $a_n = 1$, and we clearly have $\lim_{n\to\infty} 1 = 1 \neq 0$, whence the series $\sum_{n=0}^{\infty}$.

EXAMPLE 5.5.4. In Example 5.3.5 we saw that the series $\sum_{n=0}^{\infty}(-1)^n$, of general term $a_n = (-1)^n$, diverges. Once again, this can also be deduced from Theorem 5.5.1: since $\lim_{n\to\infty}(-1)^n$ does not exist, in other words the sequnece $\{(-1)^n\}_{n=0}^{\infty}$ diverges, we deduce that the series $\sum_{n=0}^{\infty}(-1)^n$ diverges.

EXAMPLE 5.5.5. The series

$$\sum_{n=0}^{\infty} \frac{n^2 + 3n}{2n^2 - n}$$

diverges. Its general term is $a_n = \frac{n^2+3n}{2n^2-n}$, and via a growth-rate argument (see Section 3.2 and the last assertion in Proposition 5.1.7) we easily compute

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2 + 3n}{2n^2 - n} = \frac{1}{2} \neq 0;$$

therefore, the series diverges by Theorem 5.5.1.

The divergence test, as indicated by the name, only provides sufficient conditions for the divergence of a series: if $\lim_{n\to\infty} a_n$ does not exist ,or if it exists but is not equal to 0, then the series $\sum_{n=0}^{\infty} a_n$ diverges necessarily. However, if $\lim_{n\to\infty} a_n = 0$, nothing can be inferred about the convergence/divergence of the series. For instance, when r is a real number with $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, and we have already seen in Section 5.4 that also the series$\sum_{n=0}^{\infty} r^n$ converges. However, the following fundamental example shows that there exist divergent series $\sum_{n=0}^{\infty} a_n$ satisfying $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 5.5.6 (The harmonic series). The harmonic series is the series

$$\sum_{n=1} \frac{1}{n},$$

[7]This is a fact worth highlighting: the sequence $\{S_N\}_{N=0}^{\infty}$ of partial sums of a series $\sum_{n=0}^{\infty} a_n$ allows to completely reconstruct the original sequence $\{a_n\}_{n=0}^{\infty}$ via the formula

$$a_n = S_n - S_{n-1} \quad \text{for all } n \geq 0,$$

with the convention that $S_{-1} = 0$.

obtained by summing all inverses of natural numbers. We clearly have $\lim_{n\to\infty} \frac{1}{n} = 0$. However, we claim that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Observe that the following inequalities hold:

$$1 + \left(\frac{1}{2} + \frac{1}{3}\right) > 1 + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2}\,,$$

$$1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2}\,,$$

$$\begin{aligned} 1 &+ \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15}\right) \\ &> 1 + \frac{1}{2} + \frac{1}{2} + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\,. \end{aligned}$$

Iterating the previous steps, we arrive at the following inequality, valid for all integers $N \geq 1$:

$$\sum_{n=1}^{1+2+\cdots+2^N} \frac{1}{n} = \sum_{n=1}^{2^{N+1}-1} \frac{1}{n} > 1 + \frac{N}{2}\,.$$

As $\lim_{N\to\infty} 1 + \frac{N}{2} = +\infty$, from the last displayed inequality we deduce that the sequence of partial sums for the harmonic series is not bounded; by the fourth property in Proposition 5.3.6, we conclude that the harmonic series cannot converge.

5.5.2. The integral test. The integral test is useful to determine the behaviour of series with terms defined by functions whose indefinite integral from 1 to $+\infty$ is known. For instance, we have seen in Example 5.5.6 that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. What about the series $\sum_{n=1} \frac{1}{n^2}$? As n^2 grows faster than n, $\frac{1}{n^2}$ tends to 0 more rapidly than $\frac{1}{n}$, and this might play in favour of convergence. In fact, the following theorem tells us that the behaviour of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is dictated by the behaviour of the indefinite integral $\int_1^{+\infty} \frac{1}{x^2}\,\mathrm{d}x$.

Theorem 5.5.7 (Integral test). *Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence defined by $a_n = f(n)$ where $f\colon [1, +\infty) \to \mathbb{R}$ is a function satisfying the following properties:*

(1) f is continuous;
(2) $f(x) \geq 0$ for all $x \geq 1$;
(3) f is non-increasing, that is, $x \leq y$ implies $f(x) \geq f(y)$ for all $x, y \geq 1$.

Then the following assertions hold:

(1) if the improper integral $\int_1^{+\infty} f(x)\,\mathrm{d}x$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges;
(2) if the improper integral $\int_1^{+\infty} f(x)\,\mathrm{d}x$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 5.5.8 (The series $\sum_n n^{-p}$). Equipped with the comparison and the integral tests we can fully determine the behaviour of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where p is a real number. We already know that the series diverges for $p = 1$, as it reduces to the harmonic series in this case (see Example 5.5.6). Now when $p \leq 1$, we have that $n^p \leq n$, and thus $\frac{1}{n^p} \geq \frac{1}{n}$. It follows by Theorem 5.5.10 and the divergence of the harmonic series that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges to infinity for all $p \leq 1$. Observe that for $p \leq 0$ this could also be deduced by Theorem 5.5.1, as n^{-p} does not tend to zero in this case.

It remains to analyze the case $p > 1$, where both the divergence and the comparison test (comparing the given series with the harmonic series) are inconclusive. Here, however, we can successfully applied the integral test, Theorem 5.5.7. Indeed, we have $n^{-p} = f(n)$ for $f(x) = x^{-p}$, which is a continuous function $[1, +\infty) \to \mathbb{R}$, satisfying $f(x) \geq 0$ for all $x \geq 1$ and

$f(y) \leq f(x)$ for all $1 \leq x \leq y$. Thus f fulfils the assumptions of Theorem 5.5.7, which means that the behaviour of $\sum_{n=0}^{\infty} n^{-p}$ is dictated by the behaviour of $\int_1^{+\infty} x^{-p}\, \mathrm{d}x$. We compute

$$\int_1^{+\infty} \frac{1}{x^p}\, \mathrm{d}x = \lim_{t \to +\infty} \int_1^t \frac{1}{x^p}\, \mathrm{d}x = \lim_{t \to +\infty} \frac{1}{1-p} x^{1-p} \Big|_{x=1}^{x=t} = \frac{1}{p-1} ,$$

where the last equality follows from the fact that $\lim_{t \to +\infty} t^{1-p} = 0$ as $p > 1$. Since the integral converges, we deduce that $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.

We summarize the results: the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a real number,

- converges for $p > 1$ and
- diverges for $p \leq 1$.

Example 5.5.9. Let's examine the behaviour of the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} .$$

Observe that the most direct applications of the comparison test are inconclusive: indeed, $\frac{1}{n \log n} \leq \frac{1}{n}$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and on the other hand $\frac{1}{n \log n} \geq \frac{1}{n^p}$ for every $p > 1$ and for all sufficiently large n (where the precise quantification of "sufficiently large" depends on p) but $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

However, we can observe that $\frac{1}{n \log n} = f(n)$ for the function $f \colon [2, +\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x \log x}$, which is continuous, non-negative and non-increasing. By Theorem 5.5.7, we only need to verify whether $\int_2^{+\infty} \frac{1}{x \log x}\, \mathrm{d}x$ converges or diverges in order to determine the behaviour of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$. Since $\log \log x$ is an anti-derivative of $\frac{1}{x \log x}$, we have

$$\int_2^{+\infty} \frac{1}{x \log x}\, \mathrm{d}x = \lim_{t \to +\infty} \int_2^{+\infty} \frac{1}{x \log x}\, \mathrm{d}x = \lim_{t \to +\infty} \log \log x \Big|_{x=2}^{x=t} = \lim_{t \to +\infty} \log \log t - \log \log 2 = +\infty ,$$

so that we conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} = +\infty .$$

5.5.3. The comparison test. A beneficial way of determining the behaviour of a series is to relate it to the behaviour of a similar-looking, already known series. A first way to implement this strategy is to compare terms of two series via inequalities.

Theorem 5.5.10 (Comparison test). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences such that $0 \leq a_n \leq b_n$ for all $n \geq 0$.*

(1) If $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
(2) If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.

Proof. The second statement is the contrapositive of the first one, whence it suffices to show the validity of the latter. Suppose therefore that $0 \leq a_n \leq b_n$ for all $n \geq 0$, and that $\sum_{n=0}^{\infty} b_n$ converges. By definition, this means that the sequence $\{S_N\}_{N=0}^{\infty}$ of partial sums $S_N = \sum_{n=0}^{N} b_n$ converges. By Theorem 5.2.5, this implies that $\{S_N\}_{N=0}^{\infty}$ is bounded: in particular, there is $M \in \mathbb{R}$ such that $S_N \leq M$ for all $N \geq 0$. As $a_n \leq b_n$ for all $n \geq 0$, we deduce that the partial sums $S'_N = \sum_{n=0}^{N} a_n$ form a sequence which is bounded from above: for all $N \geq 0$,

$$S'_N = \sum_{n=0}^{N} a_n \leq \sum_{n=0}^{N} b_n = S_N \leq M .$$

Furthermore, the fact that $a_n \geq 0$ implies that $S'_0 \leq S'_1 \leq S'_2 \leq \cdots S'_N \leq \cdots$. By Theorem 5.2.7, we infer that the sequence $\{S'_N\}_{N=0}^{\infty}$ converges, which by definition means that $\sum_{n=0}^{\infty} a_n$ converges. This finishes the proof. $\square$

Example 5.5.11. Let us determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3+8}$$

converges or diverges. We compare it with the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$. We have $n^3+8 > n^3 > 0$, whence $0 < \frac{1}{n^3+8} < \frac{1}{n^3}$, for all $n \geq 1$. As $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (see Example 5.5.8), so does the series $\sum_{n=1}^{\infty} \frac{1}{n^3+8}$ by Theorem 5.5.10.

Example 5.5.12. We would like to understand the behaviour of the series

$$\sum_{n=2}^{\infty} \frac{1}{\log n} .$$

As $\log n$ grows much more slowly than n, it is useful to compare the given series to the harmonic series. More precisely, We know that $0 < \log n < n$, whence $0 < \frac{1}{n} < \frac{1}{\log n}$ for all $n \geq 2$; as $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (see Example 5.5.6), so does the series $\sum_{n=2}^{\infty} \frac{1}{\log n}$ by Theorem 5.5.10.

Example 5.5.13. We study the series

$$\sum_{n=1}^{\infty} \frac{8+\sin(n^2)}{\sqrt{n}} .$$

As $\sin x \geq -1$ for all $x \in \mathbb{R}$ we have

$$\frac{8+\sin(n^2)}{\sqrt{n}} \geq \frac{7}{\sqrt{n}} ;$$

the series $\sum_{n=1}^{\infty} \frac{7}{\sqrt{n}}$ diverges, from Example 5.5.8 and the first assertion in Proposition 5.3.6. Theorem 5.5.10 thus implies that the series $\sum_{n=1}^{\infty} \frac{8+\sin(n^2)}{\sqrt{n}}$ diverges.

5.5.4. The limit comparison test. A second, more precise way to relate the behaviour of two series consists in comparing their terms asymptotically.

Theorem 5.5.14 (Limit comparison test). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences such that $a_n \geq 0$, $b_n \geq 0$ for all $n \geq 0$. Suppose that there exists a real number $L > 0$ such that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L .$$

Then the following assertions hold:

(1) if the series $\sum_{n=0}^{\infty} b_n$ converges, then so does the series $\sum_{n=0}^{\infty} a_n$;
(2) if the series $\sum_{n=0}^{\infty} b_n$ diverges, then so does the series $\sum_{n=0}^{\infty} a_n$.

Therefore, under the assumptions of the previous theorem, the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ have exactly the same asymptotic behaviour. Beware that the limit must be a non-zero positive real number for the result to hold.

Proof. It follows rather directly from Theorem 5.5.10. Assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L ,$$

where $L > 0$ is a real number. We start with the case when $\sum_{n=0}^{\infty} b_n$ converges. By definition of limit, there exists an integer $N \geq 1$ such that, for all $n \geq N$,

$$\frac{a_n}{b_n} \leq L+1 , \quad \text{that is,} \quad a_n \leq (L+1)b_n .$$

As the series $\sum_{n=N}^{\infty} b_n$ converges, so does the series $\sum_{n=N}^{\infty}(L+1)b_n$ by the first assertion in Proposition 5.3.6. It follows from Theorem 5.5.10 that the series $\sum_{n=N} a_n$ converges as well, and hence so does the series $\sum_{n=0}^{\infty} a_n$, obtained from the former by adding only finitely many terms.

Assume now that $\sum_{n=0}^{\infty} b_n$ diverges. By definition of limit there exists a real number $0 < L' < L$ and an integer $M \geq 1$ such that, for all $n \geq M$,

$$\frac{a_n}{b_n} \geq L' \,, \quad \text{that is,} \quad a_n \geq L' b_n \,.$$

As $\sum_{n=M}^{\infty} b_n$ diverges, so does the series $\sum_{n=M}^{\infty} L' b_n$ by the first assertion in Proposition 5.3.6. As a consequence of Theorem 5.5.10, we conclude that the series $\sum_{n=M}^{\infty} a_n$ must diverge, and *a fortiori* the same holds for the series $\sum_{n=0}^{\infty} a_n$. $\square$

EXAMPLE 5.5.15. We examine the behaviour of the series

$$\sum_{n=1}^{\infty} \frac{n^2+5n+8}{5n^3+8n+9} \,.$$

Asymptotically, the sequence $\{\frac{n^2+5n+8}{5n^3+8n+9}\}_{n=1}^{\infty}$ should behave like the sequence $\{\frac{n^2}{5n^3}\}_{n=1}^{\infty} = \{\frac{1}{5n}\}_{n=1}^{\infty}$, so that we would like to compare the given series with the series $\sum_{n=1}^{\infty} \frac{1}{5n}$, which diverges in view of Example 5.5.6 and the first assertion in Proposition 5.3.6. In fact, we have

$$\lim_{n\to\infty} \frac{\frac{n^2+5n+8}{5n^3+8n+9}}{\frac{1}{5n}} = \lim_{n\to\infty} \frac{5n^3+25n^2+30n}{5n^3+8n+9} = 1 \,,$$

so that by Theorem 5.5.14 we deduce that the series $\sum_{n=1}^{\infty} \frac{n^2+5n+8}{5n^3+8n+9}$ diverges.

EXAMPLE 5.5.16. Let us show that the series

$$\sum_{n=1}^{\infty} \frac{3n+1}{n^3+2}$$

converges by means of Theorem 5.5.14. It is apparent that we can profitably compare the given series with $\sum_{n=1}^{\infty} \frac{1}{n^2}$: indeed,

$$\lim_{n\to\infty} \frac{\frac{3n+1}{n^3+2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^3+n^2}{n^3+2} = 3 \,.$$

As the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by Example 5.5.8, we derive from Theorem 5.5.14 that the series $\sum_{n=1}^{\infty} \frac{3n+1}{n^3+2}$ converges as well.

5.5.5. The alternating series test. Up to this point we have only discussed (except for the divergence test) methods of proving convergence or divergence for series with non-negative terms, namely those series $\sum_{n=0}^{\infty} a_n$ with $a_n \geq 0$ for all $n \geq 0$. We shall now discuss some criteria to determine the behaviour of series with general term of possibly changing sign.

The first criterion concerns series with *alternating signs*.

DEFINITION 5.5.17 (Alternating series). A series $\sum_{n=1} a_n$ is called **alternating** if $a_n = (-1)^n |a_n|$ for all $n \geq 1$ or $a_n = (-1)^{n-1}|a_n|$ for all $n \geq 1$.

Unravelling the definition, an alternating series has the property that its terms a_n alternate between being positive and negative, that is, either

$$a_1 \geq 0 \,, \quad a_2 \leq 0 \,, \quad a_3 \geq 0 \,, \quad a_4 \leq 0 \,, \quad \ldots$$

or

$$a_1 \leq 0 \,, \quad a_2 \geq 0 \,, \quad a_3 \leq 0 \,, \quad a_4 \geq 0 \,, \quad \ldots \,.$$

For alternating series, we have the following useful convergence test.

THEOREM 5.5.18 (Alternating series). *Let*

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

be an alternating series, where $b_n \geq 0$ *for all* $n \geq 1$. *Suppose that*

(1) $b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_n \geq \cdots$ *and*
(2) $\lim_{n\to\infty} b_n = 0$.

Then the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *converges.*

REMARK 5.5.19. Notice that Theorem 5.5.18 can be seen as a partial converse to Theorem 5.5.1. We have discusses that, in general, convergence of a series $\sum_{n=1}^{\infty} a_n$ necessitates that $\lim_{n\to\infty} a_n = 0$, whereas the latter condition does not ensure convergence of the series. For alternating series $a_n = (-1)^{n-1}b_n$, however, if we add the monotonicity assumption $b_1 \geq b_2 \geq \cdots \geq b_n \geq \cdots$, then the series is guaranteed to converge.

EXAMPLE 5.5.20. Let's examine the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} .$$

It can be expressed as in Theorem 5.5.18 with $b_n = \frac{1}{n} > 0$. The sequence $\{b_n\}_{n=1}^{\infty}$ is decreasing and converges to 0, hence the assumptions of Theorem 5.5.18 are satisfied: therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{b_n}$ converges.

EXAMPLE 5.5.21. We turn to the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{3n^2+5}}{n+1} .$$

We can express it as in Theorem 5.5.18 with $b_n = \frac{\sqrt{3n^2+5}}{n+1} > 0$. However, an elementary growth-rates argument shows that

$$\lim_{n\to\infty} \frac{\sqrt{3n^2+5}}{n+1} = \sqrt{3} \neq 0 ,$$

whence the given series cannot converge, by Theorem 5.5.1.

Error bound for alternating series. An important feature of alternating series satisfying the assumptions of Theorem 5.5.18 is that it is possible to give an explicit upper bound for the error we commit when approximating their limit value with partial sums, a feature which is relevant in applications.

THEOREM 5.5.22 (Error bound for alternating series). *Let*

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$

be an alternating series satisfying the assumptions of Theorem 5.5.18. Let L be its sum. For every integer $N \geq 1$, the following inequality holds:

$$\left|\sum_{n=1}^{N}(-1)^{n-1}b_n - L\right| \leq b_{N+1} .$$

Hence the gap between the N-th partial sum of the series and its limiting value does not exceed the $(N+1)$-th term of the sequence $\{b_n\}_{n=1}^{\infty}$.

PROOF. From the proof of Theorem 5.5.18, we have for each odd N that

$$b_{N+1} = S_N - S_{N+1} = S_N - L + L - S_{N+1} \geq S_N - L = |S_N - L| ,$$

where $S_N = \sum_{n=1}^{N}(-1)^{n-1}b_n$ and the last two steps follow from the fact that $S_{N+1} \leq L$ and $S_N \geq L$. On the other hand, when N is even, we have

$$b_{N+1} = S_{N+1} - S_N = S_{N+1} - L + L - S_N \geq L - S_N = |S_N - L| ,$$

where the last two steps follow from the fact that $S_{N+1} \geq L$ and $S_N \leq L$. The proof is concluded. $\square$

Example 5.5.23. The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges, as shown in Example 5.5.20. Let L denote its sum, and suppose we would like to find some integer $N \geq 1$, as small as possible, such that we can approximate L with the finite sum $\sum_{n=1}^{N} \frac{(-1)^n}{n}$ within an error of at most 10^{-10}. From Theorem 5.5.22, we know that, for all $N \geq 1$,

$$\left|\sum_{n=1}^{N}(-1)^{n-1}b_n - L\right| = \left|\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} - L\right| \leq b_{N+1} = \frac{1}{N+1} \ ;$$

hence, imposing the condition $\frac{1}{N+1} \leq 10^{-10}$, that is, $N \geq 10^{10} - 1$, ensures that

$$\left|\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} - L\right| \leq 10^{-10} \ .$$

Notice that, despite it sounding counterintuitive at first, we don't need to know the exact value of L to find such an approximation.

Example 5.5.24. The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}$$

satisfies the assumptions of Theorem 5.5.18, as it is straightforward to verify. Call $L = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}$. We would like to find an integer $N \geq 1$, as small as possible, such that the finite sum $\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^4}$ approximates L within an error not exceeding 10^{-1}. By Theorem 5.5.22 we know that, for every $N \geq 1$,

$$\left|\sum_{n=1}^{N}(-1)^{n-1}b_n - L\right| = \left|\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^4} - L\right| \leq b_{N+1} = \frac{1}{(N+1)^4} \ ,$$

so that requiring

$$\frac{1}{(N+1)^4} \leq 10^{-1} \ , \quad \text{that is,} \quad (N+1)^4 \geq 10$$

does the job. Since $1^4 = 1 < 10 < 16 = 2^4$, we see that taking $N = 1$ already suffices to have the desired control on the error.

Example 5.5.25. It turns out that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} = \log 3 - \log 2 \ ,$$

a fact which we shall be able to properly justify with the content developed in forthcoming sections. For the moment, it suffices to know that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}$ converges as it satisfies the assumptions of Theorem 5.5.18. Let's try to find an integer $N \geq 1$, as small as possible, such that we can approximate $\log 3 - log 2$ with a finite sum $\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n2^n}$ with an error not exceeding 10^{-3}. From Theorem 5.5.22 we know that, for all $N \geq 1$,

$$\left|\sum_{n=1}^{N}(-1)^{N-1}b_n - L\right| = \left|\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n2^n} - (\log 3 - \log 2)\right| \leq b_{N+1} = \frac{1}{(N+1)2^{N+1}} \ ;$$

therefore, it suffices to impose the condition

$$\frac{1}{(N+1)2^{N+1}} \leq 10^{-3} \ , \quad \text{that is,} \quad (N+1)2^{N+1} \geq 10^3 \ ;$$

we check manually that $7 \cdot 2^7 = 7 \cdot 128 = 896 < 10^3$, whereas $8 \cdot 2^8 = 2^{11} = 2048 > 10^3$, so that $N = 7$ serves our purposes.

5.5.6. Absolute and conditional convergence. We proceed with our study of series $\sum_{n=1}^{\infty} a_n$ where the terms a_n are possibly of changing sign. Alternating series, which we have examined in the previous section, are only a special instance of such series, where the signs alternate between $+$ and $-$. In order to investigate the convergence properties of more general series with terms arbitrary sign, we need to develop new tools. The main one is arguably the notion of absolute convergence, which we shall shortly see is stronger than convergence, and reduces the study to series with non-negative terms.

Definition 5.5.26 (Absolute and conditional convergence). A series $\sum_{n=1}^{\infty} a_n$ is called

(1) **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ converges;
(2) **conditionally convergent** if it converges but the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

Observe that the two properties are mutually exclusive, namely they cannot hold at the same time. For a series $\sum_{n=1}^{\infty} a_n$ with non-negative terms, that is, with $a_n \geq 0$ for all $n \geq 1$, the concept of absolute convergence coincides with the one of convergence, as $|a_n| = a_n$ for all $n \geq 1$. For the same reason, such a series can never be conditionally convergent. We see thus that the two notions are suited to deal with series whose terms are of changing sign.

Example 5.5.27. Let's determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is absolutely convergent, conditionally convergent, or neither of the two. We have

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} ;$$

the harmonic series diverges, hence our original series is not absolutely convergent. It is conditionally convergent, since it converges as seen in Example 5.5.20.

Example 5.5.28. Let's find whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

is absolutely convergent, conditionally convergent, or neither of the two. We have

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3} ,$$

which is convergent as the exponent satisfies $3 > 1$. Thus, by definition, our original series is absolutely convergent. As a consequence, it is not conditionally convergent, despite being convergent itself by Theorem 5.5.18: we can write it as $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ with $b_n = \frac{1}{n^3} > 0$, and the sequence $\{b_n\}_{n=1}^{\infty}$ is thus decreasing and converging to 0.

The fundamental property of absolute convergence, which justifies *a posteriori* the terminology, is that it entails convergence.

Theorem 5.5.29. *If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.*

In loose terms, the rationale for this theorem is that when the series $\sum_{n=1}^{\infty} |a_n|$, in which we constantly add non-negative terms, converges, then the same ought to be true for the original series $\sum_{n=1}^{\infty} a_n$, in which we either add or subtract non-negative real numbers depending on the sign a_n; adding and subtracting should, in a sense, compete against each other and produce cancellations, so that the final sum should *a fortiori* converge.

PROOF. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Since $-|a_n| \leq a_n \leq |a_n|$ for all $n \geq 1$, we may add $|a_n|$ to all the terms of the latter inequality to obtain

$$0 \leq a_n + |a_n| \leq 2|a_n| \quad \text{for all } n \geq 1. \tag{5.5.1}$$

As the series $\sum_{n=1}^{\infty} |a_n|$ converges by assumption, so does the series $\sum_{n=1}^{\infty} 2|a_n|$ by the first assertion in Proposition 5.3.6. It now follows from (5.5.1) and Theorem 5.5.10 that the series

$$\sum_{n=1}^{\infty} a_n + |a_n|$$

converges. In light of the second assertion in Proposition 5.3.6, we deduce that, since $a_n = a_n + |a_n| - |a_n|$ and both the series $\sum_{n=1}^{\infty} a_n + |a_n|$ and $\sum_{n=1}^{\infty} |a_n|$ converge, so does the series $\sum_{n=1}^{\infty} a_n$, which finishes the proof. □

The upshot of the previous theorem is that, if we want to study the behaviour of a general series $\sum_{n=1}^{\infty} a_n$, the first step is to investigate the series $\sum_{n=1}^{\infty} |a_n|$, which has non-negative terms and as such is amenable to be studied with all the criteria developed so far, with the exception of Theorem 5.5.18, and with all those we shall subsequently develop. If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then by definition the original series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and from the last theorem we also infer that $\sum_{n=1}^{\infty} a_n$ converges. If, on the other hand, the series $\sum_{n=1}^{\infty} |a_n|$ diverges, then this fact by itself doesn't tell us anything about the behaviour of the original series $\sum_{n=1}^{\infty} a_n$, which therefore needs to be studied by other means, for instance Theorem 5.5.1 or Theorem 5.5.18.

To sum up, a series $\sum_{n=1}^{\infty} a_n$ is either

- absolutely convergent,
- or conditionally convergent,
- or divergent,

the three possibilities being mutually exclusive. The second possibility presents itself only for series with terms of changing sign.

EXAMPLE 5.5.30. Let's examine whether the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

converges. Observe that Its general term $\frac{\sin n}{n^2}$ can be either positive or negative, depending on n, and it cannot be written in a form compatible with Theorem 5.5.18 since the sign is not alternating. The only criteria we have to determine the convergence of such a series are thus Theorem 5.5.1 and Theorem 5.5.29. We have $\lim_{n\to\infty} \frac{\sin n}{n^2} = 0$ by comparison with the sequences $\{-\frac{1}{n^2}\}$ and $\{\frac{1}{n^2}\}$. Thus, Theorem 5.5.1 tells us nothing. However, Theorem 5.5.29 can be applied here. We have

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2},$$

and since $|\sin(n)| \leq 1$ for all n we can compare the latter series to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges. Thus, by Theorem 5.5.10, the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely and thus, by Theorem 5.5.29, it converges.

5.5.7. Optional: rearrangements of terms in infinite series. A *rearrangement* of a sequence $\{a_n\}_{n=1}^{\infty}$ is a sequence $\{b_n\}_{n=1}^{\infty}$ which is obtained by permuting the terms a_n; in mathematically precise terms, there is a bijective map $\phi\colon \{1, 2, 3, \ldots, n, \ldots\} \to \{1, 2, 3, \ldots, n, \ldots\}$ such that $b_n = a_{\phi(n)}$ for all $n \geq 1$.

THEOREM 5.5.31. *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Then there is a real number L such that, for every rearrangement $\{a_{\phi(n)}\}_{n=1}^{\infty}$ of the terms of the series,*

$$\sum_{n=1}^{\infty} a_{\phi(n)} = L \,.$$

THEOREM 5.5.32. *Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series. For any real number L, there exists a rearrangement $\{a_{\phi(n)}\}_{n=1}^{\infty}$ of the terms of the series ensuring that*

$$\sum_{n=1}^{\infty} a_{\phi(n)} = L \,.$$

5.5.8. The root and ratio tests. The root test and the ratio test provide two ways of determining whether a series converges or diverges by comparing it with a geometric series.

Let us start with the ratio test, which entails examining the limiting behaviour of rations of consecutive terms of the series.

THEOREM 5.5.33 (Ratio test). *Let $\sum_{n=0}^{\infty} a_n$ be a series. Suppose that there is $\rho \geq 0$, possibly equal to $+\infty$, such that*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho \,.$$

Then the following hold:

(1) if $\rho < 1$, then the series converges absolutely;
(2) if $\rho > 1$ then the series diverges;
(3) if $\rho = 1$, then the test is inconclusive, that is, nothing can be inferred in general about the behaviour of the series.

REMARK 5.5.34. It is implicit in the assumption of existence of the limit $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ that a_n doesn't vanish for all sufficiently large integers n.

PROOF. Let's assume first that $\rho < 1$, and choose a real number ρ' satisfying $\rho < \rho' < 1$. From the definition of limit of a sequence (see Definition 5.1.4), we know that there exists an integer $N \geq 1$ such that

$$\frac{|a_{n+1}|}{|a_n|} \leq \rho' \,, \quad \text{that is,} \quad |a_{n+1}| \leq \rho' |a_n|$$

for all $n \geq N$. Iterating the previous inequality, we obtain that, for every integer $m \geq 1$,

$$|a_{N+m}| \leq \rho' |a_{N+m-1}| \leq \rho'^2 |a_{N+m-2}| \leq \cdots \leq \rho'^m |a_N| \,.$$

By comparison with the geometric series $\sum_{m=1}^{\infty} |a_N| \rho'^m$, which converges as $\rho' < 1$, we deduce from Theorem 5.5.10 that the series

$$\sum_{m=1}^{\infty} |a_{N+m}|$$

converges. Hence, so does the series

$$\sum_{n=0}^{\infty} |a_n| \,,$$

which is obtained from the previous one by adding finitely many terms $|a_0|, \ldots, |a_N|$. We have thus shown that the series $\sum_{n=0}^{\infty} a_n$ converges absolutely, as desired.

Let us now suppose that $\rho > 1$. From Definition 5.1.4 we know, in particular, that there is an integer $N \geq 1$ such that

$$\frac{|a_{n+1}|}{|a_n|} \geq 1 \,, \quad \text{that is,} \quad |a_{n+1}| \geq |a_n|$$

for all $n \geq N$. Iterating the previous inequality, we deduce that, for all integers $m \geq 1$,

$$|a_{N+m}| \geq |a_{N+m-1}| \geq |a_{N+m-2}| \geq \cdots \geq |a_N| ,$$

from which we deduce that the sequence $\{|a_n|\}_{n=0}^{\infty}$ cannot converge[8] to 0. It is straightforward to verify, from Definition 5.1.4, that a sequence $\{b_n\}_{n=0}^{\infty}$ converges to 0 if and only if the sequence $\{|b_n|\}_{n=0}^{\infty}$ converges to 0. Therefore, we infer that the sequence $\{a_n\}_{n=0}^{\infty}$ cannot converge to 0, and so the series $\sum_{n=0}^{\infty} a_n$ diverges by Theorem 5.5.1.

Finally, let us assume that $\rho = 1$, and see that there are examples showing that any behaviour is possible from the series $\sum_{n=0}^{\infty} a_n$, so that nothing can be concluded in general. Let us consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} ,$$

where p is a real number. We have that

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1$$

regardless of the value of p. However, we know from Example 5.5.8 that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$. $\square$

EXAMPLE 5.5.35. We wish to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

converges or diverges. As the denominator grows much faster than the numerator, we expect to have same behaviour of the convergent geometric series $\sum_{n=1} \frac{1}{2^n}$. Theorem 5.5.33 provides a way to formalize this intuition. We evaluate

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|\frac{(n+1)^2}{2^{n+1}}|}{|\frac{n^2}{2^n}|} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \frac{1}{2} .$$

As the latter limit is strictly smaller than 1, Theorem 5.5.33 allows us to conclude that the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges asbolutely, which is equivalent to say that it converges as it has positive terms.

EXAMPLE 5.5.36. Consider now the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} .$$

As $n!$ grows much faster than 2^n, we expect convergence. Let's prove it resorting to Theorem 5.5.33: we compute

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|\frac{2^{n+1}}{(n+1)!}|}{|\frac{2^n}{n!}|} = \lim_{n\to\infty} \frac{2n!}{(n+1)!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 .$$

Therefore, by Theorem 5.5.33 we have that the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

EXAMPLE 5.5.37. We investigate the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} .$$

[8] Unless, of course, $a_{N+m} = 0$ for all $m \geq 1$, which is excluded as pointed out in Remark 5.5.34.

Guessing the behaviour is this time problematic, as there are factorials of a similar-looking nature on both in the numerator and in the denominator. The solution is to apply again Theorem 5.5.33:

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\left|\frac{(2(n+1))!}{((n+1)!)^2}\right|}{\left|\frac{(2n)!}{(n!)^2}\right|}=\lim_{n\to\infty}\frac{(2n+2)!}{(2n)!}\left(\frac{n!}{(n+1)!}\right)^2=\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}$$
$$=\lim_{n\to\infty}\frac{4n^2+6n+2}{n^2+2n+1}=4\,.$$

The latter limit being strictly bigger than 1, Theorem 5.5.33 delivers that the series $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ diverges.

Remark 5.5.38. The presence of factorials in the general term of a series is a strong indication that Theorem 5.5.33 will provide information on the behaviour of the series: taking ratios of consecutive terms results in a lot of simplications between numerator and denominator.

We now move on to the root test, which involves examining the limiting behaviour of the n-th root of the n-th term of a series.

Theorem 5.5.39 (Root test). *Let $\sum_{n=0}^{\infty}a_n$ be a series. Suppose that there is $\rho\geq 0$, possibly equal to $+\infty$, such that*

$$\lim_{n\to\infty}|a_n|^{1/n}=\rho\,.$$

Then the following hold:

(1) if $\rho<1$, then the series converges absolutely;
(2) if $\rho>1$ then the series diverges;
(3) if $\rho=1$, then the test is inconclusive, that is, nothing can be inferred in general about the behaviour of the series.

Observe that the conclusions of Theorem 5.5.39 are entirely analogous to those of Theorem 5.5.33, except that the limits to be computed in the two theorems are different.

Proof. The proof bears a lot of resemblance to the proof of Theorem 5.5.33. Let's assume first that $\rho<1$, and choose a real number ρ' satisfying $\rho<\rho'<1$. From the definition of limit of a sequence (see Definition 5.1.4), we know that there exists an integer $N\geq 1$ such that

$$|a_n|^{1/n}\leq\rho'\,,\quad\text{that is,}\quad|a_n|\leq\rho'^n$$

for all $n\geq N$. By comparison with the geometric series $\sum_{n=N}^{\infty}\rho'^n$, which converges as $\rho'<1$, we deduce from Theorem 5.5.10 that the series

$$\sum_{n=N}^{\infty}|a_n|$$

converges. Hence, so does the series

$$\sum_{n=0}^{\infty}|a_n|\,,$$

which is obtained from the previous one by adding finitely many terms $|a_0|,\ldots,|a_{N-1}|$. We have thus shown that the series $\sum_{n=0}^{\infty}a_n$ converges absolutely, as desired.

Let us now suppose that $\rho>1$. From Definition 5.1.4 we know, in particular, that there is an integer $N\geq 1$ such that

$$|a_n|^{1/n}\geq 1\,,\quad\text{that is,}\quad|a_n|\geq 1^n=1$$

for all $n\geq N$. It follows that the sequence $\{|a_n|\}_{n=0}^{\infty}$ cannot converge to 0, and thus neither can the sequence $\{a_n\}_{n=0}^{\infty}$; therefore, the series $\sum_{n=0}^{\infty}a_n$ diverges by Theorem 5.5.1.

Lastly, assume that $\rho = 1$. Let us consider, once again, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} ,$$

where p is a real number. We have from Example 5.1.9 that

$$\rho = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^p}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^{p} = 1$$

regardless of the value of p. However, we know from Example 5.5.8 that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \le 1$. □

Example 5.5.40. We go back to the series in Example 5.5.35,

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} ,$$

and apply Theorem 5.5.39 to reprove that it is convergent. We need to evaluate

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{n^2}{2^n}\right|^{1/n} = \lim_{n\to\infty} \frac{(n^{1/n})^2}{2} = \frac{1}{2} ;$$

Theorem 5.5.33 thus implies the desired conclusion, namely that the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

Example 5.5.41. Let's study the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n^n} .$$

As n^n grows much faster than n^2, we expect the series to converge. As a matter of fact, we could readily deduce it from the previous example and Theorem 5.5.10, as $n^n \ge 2^n$, and thus $\frac{n^2}{n^n} \le \frac{n^2}{2^n}$, for all $n \ge 2$. For purposes of illustration, however, we show convergence by means of Theorem 5.5.39: we compute

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{n^2}{n^n}\right|^{1/n} = \lim_{n\to\infty} \frac{(n^{1/n})^2}{n} = 0 ,$$

so that from Theorem 5.5.39 we infer that the series $\sum_{n=1}^{\infty} \frac{n^2}{n^n}$ converges.

Example 5.5.42. Consider the series

$$\sum_{n=1}^{\infty} \frac{n^2}{e^{n^2}} .$$

As e^{n^2} grows much faster than n^2, we predict that the series shall converge. Let's apply Theorem 5.5.39 to confirm this:

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{n^2}{e^{n^2}}\right|^{1/n} = \lim_{n\to\infty} \frac{(n^{1/n})^2}{e^{n^2/n}} = \lim_{n\to\infty} \frac{(n^{1/n})^2}{e^{n}} = 0 ,$$

whence the series $\sum_{n=1}^{\infty} \frac{n^2}{e^{n^2}}$ converges.

Example 5.5.43. Let us examine the series

$$\sum_{n=1}^{\infty} \frac{(\arctan n)^n}{n^2} .$$

Observe that $\arctan n \overset{n\to\infty}{\longrightarrow} \pi/2$, hence the series above should behave like the series $\sum_{n=1}^{\infty} \frac{(\pi/2)^n}{n^2}$, which is divergent by Theorem 5.5.1 as $\frac{(\pi/2)^n}{n^2} \overset{n\to\infty}{\longrightarrow} +\infty$. A possible way to formalize this heuristics is to appeal to Theorem 5.5.14; instead, we choose to apply Theorem 5.5.39. Since

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{(\arctan n)^n}{n^2}\right|^{1/n} = \lim_{n\to\infty} \frac{\arctan n}{(n^{1/n})^2} = \frac{\pi}{2} > 1 ,$$

we deduce that the series $\sum_{n=1}^{\infty} \frac{(\arctan n)^n}{n^2}$ diverges.

Remark 5.5.44. The presence of n-th powers in the general term of a series, as in the previous four examples, indicates that Theorem 5.5.39 is likely able to detect the behaviour of the series, as taking than the $1/n$-th root of the general term will produce a lot of simplification.

5.6. Taylor polynomials

The driving theme of the rest of the chapter is the approximation of a real-valued function of a real variable by what are arguably the most elementary functions of this sort, namely polynomial functions and "infinite-degree" generalizations thereof. We shall start in this section with the theory of *local* approximation of a function by polynomials. The adjective "local" is explained by the following elementary but fundamental observation. We cannot expect to be able to approximate an arbitrary function with a polynomial function over the whole real line in any meaningful manner: for instance, we know from Section 3.2 that the exponential function e^x has a higher growth rate of any polynomial, that is,

$$e^x \gg A_0 + A_1 x + \cdots + A_n x^n$$

for any natural number $n \geq 0$ and any $A_0, \ldots, A_n \in \mathbb{R}$. Hence, for very large values of x, it is impossible to approximate e^x with any polynomial. On the other hand, if we restrict our focus to all values of x in a close proximity to a fixed value $a \in \mathbb{R}$, then such an approximation becomes possible, even with arbitrary precision. Let us begin by formalizing what we mean by approximation and "arbitrary precision".

Let us fix a function $f\colon (c,d) \to \mathbb{R}$, where $c < d \in \mathbb{R}$, and a base point $a \in (c,d)$. We also fix a natural number $n \geq 0$. The question we would like to give an answer to is the following: how well can we approximate f in a *neighborhood* of a, namely for all values of x sufficiently close to a, with a polynomial

$$A_0 + A_1(x-a) + \cdots + A_n(x-a)^n , \quad A_0, \ldots, A_n \in \mathbb{R}$$

of degree at most n? To measure the error of any approximation, namely the difference

$$E_n(x) = f(x) - (A_0 + A_1(x-a) + \cdots + A_n(x-a)^n) ,$$

we make use of a natural hierarchy provided by the polynomials $(x-a)^m$, $m \geq 1$ an integer, in the vicinity of a. We observe that, whenever $x \in \mathbb{R}$ is such that $|x-a| < 1$, then

$$|x-a| > |x-a|^2 > |x-a|^3 > \cdots > |x-a|^m > |x-a|^{m+1} > \cdots ,$$

with the discrepancy between each two consecutive terms of the previous chain of inequalities increasing as x approaches a: indeed, for every integer $m \geq 1$,

$$\lim_{x\to a} \frac{|x-a|^m}{|x-a|^{m+1}} = \lim_{x\to a} \frac{1}{|x-a|} = +\infty ,$$

so that $|x-a|^m$ becomes much larger than $|x-a|^{m+1}$ as x approaches a. Now, as we are free to choose our approximating polynomial, provided its degree remains bounded by n, it is desirable to have a quality of the approximation that is better than the smallest "monomial" (centered at a) $A_n(x-a)^n$ appearing in the polynomial. To be precise, we aim to have

$$\lim_{x\to a} \frac{E_n(x)}{(x-a)^n} = 0 ,$$

so that the error is becomes much smaller than $(x-a)^n$ as x approaches a.

Let us proceed step by step, starting with the case $n = 0$. We would like to find the constant polynomial A_0 which best approximates $f(x)$ for x close to a, according to the rule dictated above: we want to find $A_0 \in \mathbb{R}$ such that

$$\lim_{x\to a} E_0(x) = \lim_{x\to a} f(x) - A_0 = 0 .$$

Henceforth, *we assume that the function is continuous* on (c, d), so that $\lim_{x\to a} f(x) = f(a)$. Therefore, not surprisingly, we must have

$$A_0 = f(a) ,$$

and thus for a continuous f there is a unique constant polynomial which best approximates $f(x)$ for x near a, namely the constant polynomial $f(a)$.

Let's now look at the case $n = 1$: here we need to find the best *linear approximation* of f near a, namely we need to find $A_0, A_1 \in \mathbb{R}$ such that

$$\lim_{x\to a} \frac{E_1(x)}{(x-a)} = \lim_{x\to a} \frac{f(x) - A_0 - A_1(x-a)}{x-a} = 0 . \tag{5.6.1}$$

First, we observe that, if the previous condition is to be satisfied, then *a fortiori* the numerator of the last displayed fraction must satisfy

$$\lim_{x\to a} f(x) - A_0 - A_1(x-a) = 0 ,$$

since the denominator $x - a$ tends to 0 as x tends to a. As

$$\lim_{x\to a} f(x) - A_0 - A_1(x-a) = f(a) - A_0 ,$$

we must have $A_0 = f(a)$. From (5.6.1) we thus obtain the condition

$$0 = \lim_{x\to a} \frac{f(x) - f(a) - A_1(x-a)}{x-a} = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} - A_1 , \tag{5.6.2}$$

where we observe that it is possible to simplify the fraction $\frac{A_1(x-a)}{x-a}$ in the last step as the limit, by definition, only depends on the values $x \neq a$ which are sufficiently close to a. Assume hereinafter that f is differentiable on (c, d). By definition of derivative (see Definition B.1.1 in the Appendix),

$$\lim_{x\to a} \frac{f(x) - f(a)}{x-a} = f'(a) .$$

Therefore, the condition in (5.6.2) amounts to $A_1 = f'(a)$. Therefore, for a differentiable f, there is a unique polynomial of degree at most 1 which best approximates $f(x)$ for x near a, namely the linear polynomial $f(a) + f'(a)(x-a)$.

Remark 5.6.1. This has been introduced in Calculus I as the *linearization* of a differentiable function f at $x = a$. Its graph is the straight line that best approximates $f(x)$ for values of x near a.

We proceed with the case $n = 2$: we want to find the best *quadratic approximation* of f near a, that is, we wish to find A_0, A_1 and $A_2 \in \mathbb{R}$ such that

$$\lim_{x\to a} \frac{E_2(x)}{(x-a)^2} = \lim_{x\to a} \frac{f(x) - A_0 - A_1(x-a) - A_2(x-a)^2}{(x-a)^2} = 0 . \tag{5.6.3}$$

If the previous condition is to be satisfied, then *a fortiori* the numerator must tend to 0 as x tends to a, and so must also when divided by the linear factor $(x - a)$. Retracing the steps above, such two conditions are easily seen to lead to $A_0 = f(a)$, $A_1 = f'(a)$. It remains to find A_2, rewriting (5.6.3) as

$$0 = \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)^2} - A_2 . \tag{5.6.4}$$

Observe that evaluating the limit of

$$\frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)^2}$$

as x tends to a by direct substitution leads to the indeterminate form $0/0$. Let's apply Theorem 3.1.3 and investigate, instead,

$$\lim_{x\to a} \frac{(f(x) - f(a) - f'(a)(x-a))'}{((x-a)^2)'} = \lim_{x\to a} \frac{f'(x) - f'(a)}{2(x-a)} .$$

Again, we are left with an indeterminate form $0/0$, to eliminate which we apply Theorem 3.1.3 once more, assuming that f is twice differentiable on (c, d):

$$\lim_{x\to a} \frac{(f'(x) - f'(a))'}{(2(x-a))'} = \lim_{x\to a} \frac{f''(x)}{2} = \frac{f''(a)}{2} ,$$

where the last step follows by assuming that f'' is continuous on (c, d). We deduce from Theorem 3.1.3 that, when f'' exists and is continuous on (c, d),

$$\lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)^2} = \frac{f''(a)}{2} ,$$

which together with the condition (5.6.4) yields $A_2 = \frac{f''(a)}{2}$. Therefore, we have established that the best quadratic approximation of f near a is given by the polynomial $f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x-a)^2$.

We now enunciate and prove the general result.

THEOREM 5.6.2. *Let $f\colon (c, d) \to \mathbb{R}$ be a function, $a \in (c, d)$ a point. Let also $n \geq 0$ be an integer. Assume that f is n-times differentiable, and that the n-th derivative $f^{(n)}$ is continuous on (c, d). Then there exists a unique polynomial $T_{n,f,a}(x)$ of degree at most n satisfying*

$$\lim_{x\to a} \frac{f(x) - T_{n,f,a}(x)}{(x-a)^n} = 0 ; \tag{5.6.5}$$

it is given explicitly by

$$T_{n,f,a}(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n . \tag{5.6.6}$$

Furthemore, the polynomial $T_{n,f,a}(x)$ can be equivalently characterized as the unique polynomial of degree at most n such that

$$f(a) = T_{n,f,a}(a) , \quad f'(a) = T'_{n,f,a}(a) , \quad \ldots \quad , \quad f^{(n)}(a) = T^{(n)}_{n,f,a}(a) .$$

PROOF. In the discussion preceding the statement of the theorem, we have already shown the first assertion for the cases $n = 0$, $n = 1$ and $n = 2$. Let us now show that it holds for an arbitrary integer $n \geq 0$. For this, we assume that it holds for all integers $0 \leq k \leq n$ up to a fixed integer n, and show that it then must hold also for the successive integer $n + 1$; this ensures[9] that it holds for all $n \geq 0$. Assume thus that f is $(n+1)$-times differentiable on (c, d),

[9]This is called the *principle of mathematical induction*, which allows to rigorously prove many statements which depend on an integer $n \geq 0$. It works as follows: if $P(n)$ is a certain assertion which involves an integer $n \geq 0$, then in order to prove that $P(n)$ holds true for all $n \geq 0$ it suffices to show that $P(0)$ holds true, namely that the property is true in the case $n = 0$, and then that, whenever $P(n)$ holds true for a a given fixed integer $n \geq 0$, then it must hold true also for the successive integer $n + 1$. In this way, we know for instance that $P(1)$ holds true as $P(0)$ has been previously shown to hold; but then $P(2)$ holds true as well, and therefore $P(3)$ and so forth. It is possible to continue this chain of deductions indefinitely. Rigorously speaking, the validity of such a principle rests on the following fundamental property of the set $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ of natural numbers: if S is a subset of $\mathbb{N}$ with the property that $0 \in S$ and that $n + 1 \in S$ whenever $n \in S$, then $S = \mathbb{N}$; in other words, every set of natural numbers containing 0 and the successor of each of its elements must be equal to the whole set of natural numbers.

We will not introduce the principle of mathematical induction formally in this course, but resort to it in the course of the present proof.

and that $f^{(n+1)}$ is continuous on (c,d). Let also $A_0+A_1(x-a)+A_2(x-a)^2+\cdots A_{n+1}(x-a)^{n+1}$ be an arbitrary polynomial of degree at most n; we first need to show that there exists at least one choice of the coefficients $A_0, A_1, \ldots, A_{n+1}$ ensuring that

$$\lim_{x\to a} \frac{f(x)-(A_0+A_1(x-a)+\cdots A_{n+1}(x-a)^{n+1})}{(x-a)^{n+1}} = 0\,. \tag{5.6.7}$$

Observe that, since $\lim_{x\to a} x-a=0$, a necessary condition for (5.6.7) to hold is that

$$\lim_{x\to a} \frac{f(x)-(A_0+A_1(x-a)+\cdots A_{n+1}(x-a)^{n+1})}{(x-a)^{n}} = 0\,.$$

Now imposing such condition amounts precisely to find the best approximation of degree at most n for f near a; by assumption, we already know that the assertion we are trying to prove for $n+1$ holds true for n, so that necessarily we must have

$$A_0 = f(a)\,, \quad A_1 = f'(a)\,, \quad A_2 = \frac{f^{(2)}(a)}{2}\,, \quad \cdots\,, \quad A_n = \frac{f^{(n)}(a)}{n!}\,.$$

We only need to find the last coefficient A_{n+1}. We rewrite (5.6.7) as

$$0 = \lim_{x\to a} \frac{f(x)-(f(a)+f'(a)(x-a)+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n)}{(x-a)^{n+1}} - A_{n+1}\,. \tag{5.6.8}$$

In order to get rid of the indeterminate form $0/0$ arising from computing the limit

$$\lim_{x\to a} \frac{f(x)-(f(a)+f'(a)(x-a)+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n)}{(x-a)^{n+1}}$$

by direct substitution, we apply Theorem 3.1.3 and turn to the limit of the quotient of the derivatives, which is

$$\lim_{x\to a} \frac{f'(x)-f'(a)x-\cdots-\frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}}{(n+1)(x-a)^n}$$

and leads again to an indeterminate form $0/0$ unless $n=0$, which is a case we already dealt with. We apply Theorem 3.1.3 once more, and turn to

$$\lim_{x\to a} \frac{f''(x)-f'(a)-\cdots-\frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n-2}}{(n+1)n(x-a)^{n-1}}\,,$$

which is an indeterminate form $0/0$ unless $n=1$, which we also dealt with previously. Applying Theorem 3.1.3 $n+1$ times, it is straightforward to verify that we are left with

$$\lim_{x\to a} \frac{f^{(n+1)}(x)}{(n+1)!} = \frac{f^{(n+1)}(a)}{(n+1)!}\,,$$

where the last equality follows by continuity of the $(n+1)$-th derivative $f^{(n+1)}$. From (5.6.8) we thus deduce that

$$A_{n+1} = \frac{f^{(n+1)}(a)}{(n+1)!}\,,$$

which yields that

$$T_{n+1,f,a}(x) = f(a)+f'(a)(x-a)+\cdots+\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}$$

is the polynomial sought after, namely the one yielding the first assertion of the theorem for the case $n+1$. Observe also that our argument furnishes not only existence of a polynomial satisfying the condition (5.6.5), but also its uniqueness, as all choices of coefficients we needed to make in the process were uniquely determined.

Finally, as far as the last assertion of the theorem is concerned, it follows by elementary computations of iterated derivatives of polynomials, and from the formula in (5.6.6). $\square$

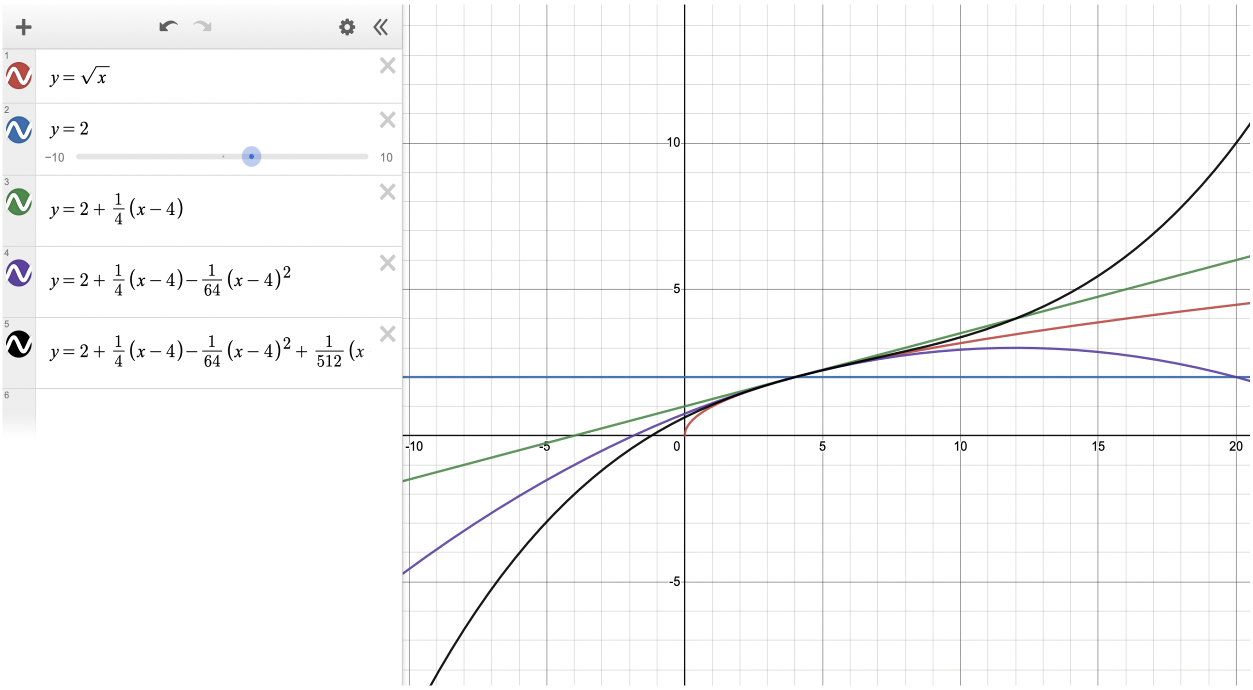

FIGURE 5.1. Graphs of the Taylor polynomials T_0, T_1, T_2, T_3 for $f(x) = \sqrt{x}$ centered at the point $x = 4$.

The unique polynomial identified by the previous theorem deserves a special name.

DEFINITION 5.6.3 (Taylor polynomial). Let $f\colon (c, d) \to \mathbb{R}$ be a function, $a \in (c, d)$ a point. Let also $n \geq 0$ be an integer, and assume that f is n-times differentiable on (c, d) and the n-th derivative $f^{(n)}$ is continuous on (c, d). We define the **n-th Taylor polynomial of f centered at a** as

$$T_{n,f,a}(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \,.$$

A point of notation: when f and a are understood from the context, we simply write $T_n(x)$ for $T_{n,f,a}$.

As established in Theorem 5.6.2, the n-th Taylor polynomial of f centered at a is the unique polynomial of degree at most n such that the difference $f(x) - T_n(x)$ is much smaller than $(x-a)^n$ as x approaches a, namely such that $\lim_{x\to a} \frac{f(x)-T_n(x)}{(x-a)^n} = 0$. We shall also say that f and T_n **agree to order n at $x = a$**. Observe that, as n increases, the quantity $(x-a)^n$ becomes smaller for x very close to a (it suffices that $|x - a| < 1$); therefore, the quality of approximation of f by its Taylor polynomial T_n near a increases as n does. Figure 5.1 illustrates this point graphically, by showing first four Taylor polynomials of the function $f(x) = \sqrt{x}$ centered at $a = 4$.

We have thus discussed how to approximate an arbitrary function, admitting sufficiently many derivatives, with a polynomial. It is natural to ask whether the procedure illustrated in this section can be carried out infinitely many times: more precisely, suppose that $f\colon (c, d) \to \mathbb{R}$ is a function admitting derivatives of any order, so that in particular each derivative $f^{(n)}$ is differentiable and thus continuous. This is the case, for instance, for the functions $f(x) = e^x$, $f(x) = \sin x$, $f(x) = \cos(x)$, $f(x)$ a polynomial, any compositions, sums and products of such functions, and many more examples. Let $a \in (c, d)$ be a point; for every integer $n \geq 0$, the n-th Taylor polynomial of f centered at a is well defined. Moreover, T_{n+1} is easily obtained from T_n by adding a single summand,

$$T_{n+1}(x) = T_n(x) + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} \,.$$

5.7. Power series

DEFINITION 5.7.1 (Power series). Given a real number c and a sequence $\{a_n\}_{n=0}^{\infty}$, the **power series** centered at c with coefficients a_n is the series

$$\sum_{n=0}^{\infty} a_n(x-c)^n \, . \quad x \in \mathbb{R}.$$

Notice the slight abuse of terminology here: by changing the value of x we obtain a family of different series, however, adhering to the common usage of the term, we refer to all of them as a single power series. We will later see that it is more appropriate to think of a power series as a *function* of the variable x, rather than as a single series.

EXAMPLE 5.7.2. Here are some examples of power series:

$$\sum_{n=0}^{\infty} \frac{1}{n!}x^n \, , \quad \sum_{n=1}^{\infty}(-1)^n(x-2)^n \, , \quad \sum_{n=0}^{\infty} 2^n n^n (x+3)^n.$$

The previous three series are centered, respectively, at 0, 2 and -3, and the sequence of coefficients are $\{\frac{1}{n!}\}_{n=0}^{\infty}$, $\{(-1)^n\}_{n=1}^{\infty}$ and $\{2^n n^n\}_{n=0}^{\infty}$. The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n(x-5)^n} \, , \quad \sum_{n=1}^{\infty} \frac{1}{nx} \, , \quad \sum_{n=1}^{\infty} \frac{1}{n^x}$$

are not power series.

Let $\sum_{n=0}^{\infty} a_n(x-c)^n$ be a given power series. For all $x \in \mathbb{R}$, we are interested in determining its behaviour, namely whether it converges absolutely, conditionally, or whether it diverges.

REMARK 5.7.3. Observe that for $x = c$ the series $\sum_{n=0} a_n(x-c)^n$ reduces to[10] a_0, as $(x-c)^n = (c-c)^n = 0$ for all $n \geq 1$; similarly, $\sum_{n=0}^{\infty} |a_n(x-c)^n| = |a_0|$. Hence, a power series always converges absolutely at the center c.

We start with two instructive examples.

EXAMPLE 5.7.4. Consider the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!}x^n \, ,$$

which is centered at 0 and has sequence of coefficients $\{\frac{1}{n!}\}_{n=0}^{\infty}$. As clarified in the previous remark, the series converges absolutely at the center $x = 0$. Suppose now x is an arbitrary non-zero real number. The presence of factorials in the sequence of coefficients suggests making use of the ratio test, Theorem 5.5.33. To this effect, we compute

$$\lim_{n\to\infty} \frac{|\frac{1}{(n+1)!}x^{n+1}|}{|\frac{1}{n!}x^n|} = \lim_{n\to\infty} \frac{n!}{(n+1)!}|x| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \, ,$$

where the last step holds for all $x \in \mathbb{R}$. Theorem 5.5.33 yields thus that the power series $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ converges absolutely for all $x \in \mathbb{R}$.

EXAMPLE 5.7.5. Let's now investigate the power series

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}(x-1)^n \, ,$$

[10] It is common in the context of power series, for a number of good reasons, to adopt the convention that $0^0 = 1$.

which is centered at 1 and has sequence of coefficients $\{\frac{1}{n2^n}\}_{n=1}^{\infty}$. Though we could equally well proceed with Theorem 5.5.33 as in the previous example, we apply instead Theorem 5.5.39 for the purpose of illustration. We evaluate

$$\lim_{n\to\infty}\left|\frac{1}{n2^n}(x-1)^n\right|^{1/n} = \lim_{n\to\infty}\frac{1}{2n^{1/n}}|x-1| = \frac{1}{2}|x-1|\ ,$$

where in the last step we applied the fact that $n^{1/n}\xrightarrow{n\to\infty}1$, as shown in Example 5.1.9. It follows from Theorem 5.5.39 that the power series $\sum_{n=1}^{\infty}\frac{1}{n2^n}(x-1)^n$ converges absolutely whenever $\frac{1}{2}|x-1|<1$, that is, when $-1<x<3$, and diverges when $\frac{1}{2}|x-1|>1$, that is, when $x>3$ and when $x<-1$. It remains to examine the cases $x=3$ and $x=-1$, when the root test is inconclusive. For $x=3$, we obtain the series

$$\sum_{n=1}^{\infty}\frac{1}{n2^n}2^n = \sum_{n=1}^{\infty}\frac{1}{n}\ ,$$

namely the harmonic series, which diverges as seen in Example 5.5.6. For $x=-1$ we get the series

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\ ,$$

which converges conditionally as shown in Example 5.5.20.

The following theorem expresses the general behaviour of a power series, which matches the one emerging from the previous two examples: there is an open interval centered at the center of the power series over which the series converges absolutely, outside of the corresponding closed interval the series diverges, and at the two boundary points of the interval all sorts of behaviour are possible.

Theorem 5.7.6. *Given a power series*

$$\sum_{n=0}^{\infty}a_n(x-c)^n\ ,$$

there exists $R\geq 0$, possibly equal to $+\infty$, such that the following hold:

(1) for any $x\in\mathbb{R}$ satisfying $|x-c|<R$, that is, $c-R<x<c+R$, the series $\sum_{n=0}^{\infty}a_n(x-c)^n$ converges absolutely;
(2) for any $x\in\mathbb{R}$ satisfying $|x-c|>R$, that is, either $x<c-R$ or $x>c+R$, the series $\sum_{n=0}^{\infty}a_n(x-c)^n$ diverges;
(3) for $x=c+R$ or $x=c-R$, nothing general can be inferred about the series $\sum_{n=0}^{\infty}a_n(x-c)^n$.

The value R given by the theorem is called the **radius of convergence** of the power series $\sum_{n=0}^{\infty}a_n(x-c)^n$. The **interval of convergence** of $\sum_{n=0}^{\infty}a_n(x-c)^n$ is the set of values $x\in\mathbb{R}$ for which the series converges. One import of the previous theorem is that such set is always an interval: If the series has radius of convergence R, then such interval can be, depending on the specific case, $(c-R,c+R)$, $[c-R,c+R)$, $(c-R,c+R]$ or $[c-R,c+R]$.

We clarify the interpretation of the statement according to the different cases $R=0$, $R=+\infty$, $0<R<+\infty$;

- if $R=0$, the statement is to be interpreted in the following way: the power series $\sum_{n=0}^{\infty}a_n(x-c)^n$ converges absolutely for $x=c$, and diverges for all $x\neq c$;
- if $R=+\infty$, the interpretation of the statement is that the power series $\sum_{n=0}^{\infty}a_n(x-c)^n$ converges absolutely for all $x\in\mathbb{R}$;
- if $R\neq 0$ and $R\neq+\infty$, then the assertion is that the power series $\sum_{n=0}^{\infty}a_n(x-c)^n$ converges absolutely for all x lying in the open interval $(c-R,c+R)$ centered at c and with radius[11] R, called **interval of convergence**, and diverges for all $x<c-R$

[11]This is where the terminology of radius of convergence comes from.

and all $x > c+R$. At the boundary points $c+R, c-R$ of the interval of convergence, the series can converge absolutely, or converge conditionally, or diverge.

Proof. We will present a proof of the first two assertions only under the following additional assumption[12]: there is $A \geq 0$, possibly equal to $+\infty$, such that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = A\,.$$

The condition is strongly reminiscent of Theorem 5.5.33, which is indeed what we are going to appeal to. Fix some $x \in \mathbb{R}$; by Remark 5.7.3 we can assume that $x \neq c$, as for $x = c$ we already know the behaviour. We compute

$$\lim_{n\to\infty} \frac{|a_{n+1}(x-c)^{n+1}|}{|a_n(x-c)^n|} = \lim_{n\to\infty} |x-c|\frac{|a_{n+1}|}{|a_n|} = |x-c| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x-c|A\,.$$

Call $R = 1/A$, with the understanding that $R = +\infty$ if $A = 0$ and that $R = 0$ if $A = +\infty$. If $|x-c|A < 1$, that is, if $|x-c| < R$, then Theorem 5.5.33 gives absolute convergence of the series $\sum_{n=0}^{\infty} a_n(x-c)^n$. If $|x-c|A > 1$, that is, if $|x-c| > R$, then Theorem 5.5.33 gives divergence of the series $\sum_{n=0}^{\infty} a_n(x-c)^n$.

Finally, consider the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n\,;$$

from Theorem 5.5.33, we know that when (assuming $x \neq 0$)

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{n+1} x^{n+1}\right|}{\left|\frac{(-1)^n}{n} x^n\right|} = \lim_{n\to\infty} \frac{n}{n+1}|x| = |x|$$

is strictly smaller than 1, then the series converges absolutely, whereas if $|x| > 1$, then the series diverges. Hence 1 is the radius of convergence of the given power series. Let's examine the situation in the critical case $|x| = 1$, that is $x = \pm 1$. For $x = 1$, we get the alternating series in Example 5.5.20, which is convergent but not absolutely convergent, that is, it's conditionally convergent. On the other hand, when $x = -1$, we get the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}\,,$$

namely the harmonic series, which diverges as seen in Example 5.5.6. Therefore, no general assertion can be made, for a given power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, at the points $c-R, c+R$, with R being the radius of convergence. □

Example 5.7.7. From Examples 5.7.4 and 5.7.5 we see that the radii of convergence of the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n\,, \quad \sum_{n=1}^{\infty} \frac{1}{n2^n}(x-1)^n$$

are, respectively, $+\infty$ and 2. The interval of convergence of the first one is the whole $\mathbb{R}$, whereas the interval of convergence of the second one is $[-1, 3)$.

Example 5.7.8. Consider the power series

$$\sum_{n=0}^{\infty} n^n x^n\,.$$

For $x \neq 0$, we apply Theorem 5.5.33 to detect its behaviour, and compute

$$\lim_{n\to\infty} \frac{|(n+1)^{n+1}x^{n+1}|}{|n^n x^n|} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n (n+1)|x| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n (n+1)|x| = +\infty\,,$$

[12] The general proof is not much more involved that the one presented here, but is omitted.

where in the last step we used the fact that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \ ,$$

as can be seen in Example 3.1.11. Hence, by Theorem 5.5.33, the power series $\sum_{n=0}^{\infty} n^n x^n$ diverges for all $x \neq 0$; by definition, this means that its radius of convergence is 0, and the interval of convergence is the degenerate interval $\{0\}$.

EXAMPLE 5.7.9. Let

$$\sum_{n=0}^{\infty} a_n (x-5)^n$$

be a power series, where the sequence of coefficients $\{a_n\}_{n=0}^{\infty}$ is unknown. Suppose we know that the given power series

- converges at $x = 8$ and
- diverges at $x = 0$.

What can we deduce about its radius of convergence R? From Theorem 5.7.6 we know that the series converges absolutely for $x \in (5-R, 5+R)$ and diverges for $x > 5+R$ or $x < 5-R$. Since the series diverges at $x = 0$, we must have $0 \leq 5-R$, where equality is allowed since the power series could potentially diverge at the boundary point $5-R$. Hence we deduce that $R \leq 5$. On the other hand, as the series converges for $x = 8$, we know that $8 \leq 5+R$, where again equality is allowed as the series could be conditionally convergent at $5+R$. Thus we infer that $R \geq 3$, so that

$$3 \leq R \leq 5 \ .$$

Nothing more precise about R can be deduced from the previous considerations.

APPENDIX A

Fundamental concepts in mathematical analysis

A.1. Properties of the real numbers

Theorem A.1.1 (Triangle inequality). *The following inequality holds for all real numbers x, y:*

$$|x+y| \leq |x| + |y| \, .$$

The inequality in the statement is called *triangle inequality* as it appeals to the geometric fact that, in each triangle, the length of a side (corresponding to the quantity $|x+y|$) cannot exceed the sum of the lengths of the other two sides (corresponding to the quantities $|x|$ and $|y|$). The geometric interpretation will be far more evident in subsequence Calculus courses, dealing with *multivariable calculus*.

A.2. Properties of functions of a real variable

Definition A.2.1 (Increasing and decreasing functions). Let E be a subset of $\mathbb{R}$. A function $f\colon E \to \mathbb{R}$ is said to be

- **increasing** if, for every $x, y \in E$, $x \leq y$ implies $f(x) \leq f(y)$, and
- **decreasing** if, for every $x, y \in E$, $x \leq y$ implies $f(x) \geq f(y)$.

Also, f is called

- **strictly increasing** if, for every $x, y \in E$, $x < y$ implies $f(x) < f(y)$, and
- **strictly decreasing** if, for every $x, y \in E$, $x < y$ implies $f(x) > f(y)$.

APPENDIX B

Differential calculus

B.1. Differentiable functions

DEFINITION B.1.1 (Derivative of a function at a point). Let $f\colon (c,d) \to \mathbb{R}$ be a function, $a \in (c,d)$ a point. We say that f is **differentiable at** a if the limit

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exists as a real number, in which case it can be equivalently expressed as

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} .$$

Such limit is called the **derivative of** f **at** a, indicated with $f'(a)$.

B.2. Derivative of products and compositions

PROPOSITION B.2.1 (Derivative of product). *Let $a < b \in \mathbb{R}$, $f, g\colon (a,b) \to \mathbb{R}$ two functions. Suppose both f and g are differentiable at a point $x_0 \in (a,b)$. Then the product function fg is differentiable at x_0, and*

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) .$$

PROPOSITION B.2.2 (Chain rule). *Let $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$, and consider two functions $f\colon (a,b) \to \mathbb{R}$, $g\colon (c,d) \to (a,b)$. If g is differentiable at a point $x_0 \in (c,d)$ and f is differentiable at the point $g(x_0)$, then the composition $f \circ g$ is differentiable at x_0, and*

$$(f \circ g)'(x_0) = f' \circ g(x_0) \cdot g'(x_0) .$$

Consequently, if g is invertible on (c,d) with inverse $g^{-1}\colon (a,b) \to \mathbb{R}$, then

$$(g^{-1})'(y_0) = \frac{1}{g'(g^{-1}(y_0))}$$

for every $y_0 \in (a,b)$.

B.3. Classical theorems of differential calculus

THEOREM B.3.1 (Mean Value Theorem). *if $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), there exists $\xi \in (a,b)$ such that*

$$f(b) - f(a) = f'(\xi)(b-a) .$$

APPENDIX C

First elements of integral calculus

C.1. The definite integral

A **partition** of a closed interval $[a,b] \subset \mathbb{R}$ is a finite collection of points $\{x_i\}_{i=0}^N$ with $a = x_0 < x_1 < \cdots < x_N = b$; in this case we say that the partition has *size* N. Given such a partition $\mathcal{P}$ of $[a,b]$, an admissible collection of *sample points* is a family $(c_i)_{i=1}^N$ of points in $[a,b]$ with the property that $c_i \in (x_{i-1}, x_i)$ for every $i = 1, \ldots, N$.

Define the length of the i-th subinterval $[x_{i-1}, x_i]$ of the partition ($1 \leq i \leq N$) as $\Delta x_i = x_i - x_{i-1}$. The *norm* of the partition $\mathcal{P}$ is defined as $\|\mathcal{P}\| = \sup_{i=1,\ldots,N} \Delta x_i$.

Given a function $f\colon [a,b] \to \mathbb{R}$, a partition $\mathcal{P} = \{x_i\}_{i=0,\ldots,N}$ of $[a,b]$ and an admissible collection $\mathcal{C} = (c_i)_{i=1,\ldots,N}$ of sample points, the corresponding **Riemann sum** of f is defined as

$$R(f, \mathcal{P}, \mathcal{C}) = \sum_{i=1}^{n} f(c_i)\Delta x_i = f(c_1)(x_1 - x_0) + \cdots f(c_N)(x_N - x_{N-1}) \;.$$

Definition C.1.1 (Definite integral, integrable function). A function $f\colon [a,b] \to \mathbb{R}$ is called (**Riemann-**) **integrable** if the limit

$$\lim_{\|\mathcal{P}\| \to 0} R(f, \mathcal{P}, \mathcal{C}) = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i$$

exists. In this case, the limit is denoted

$$\int_a^b f(x)\, \mathrm{d}x$$

and is called the **definite** (**Riemann**) **integral** of f on $[a,b]$.

When $a \leq b \in \mathbb{R}$, the definite integral $\int_b^a f(x)\, \mathrm{d}x$ of an integrable function $f\colon [a,b] \to \mathbb{R}$, where order of the limits of integration is reversed, is defined as

$$\int_b^a f(x)\, \mathrm{d}x = -\int_a^b f(x)\, \mathrm{d}x \;.$$

A large class of integrable functions[1] has been identified as in the following statement:

Theorem C.1.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a function which is continuous except at finitely many jump discontinuities. Then f is integrable.*

In particular, a continuous function $f\colon [a,b] \to \mathbb{R}$ is integrable.

Here is a list of elementary properties of the definite integral.

- If $f, g\colon [a,b] \to \mathbb{R}$ are integrable functions and $\alpha \in \mathbb{R}$, then the functions $f+g\colon [a,b] \to \mathbb{R}$ and $\alpha f\colon [a,b] \to \mathbb{R}$ are integrable; furthermore, the definite integral is linear, that is,

$$\int_a^b f(x) + g(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x + \int_a^b g(x)\, \mathrm{d}x \;, \quad \int_a^b \alpha f(x)\, \mathrm{d}x = \alpha \int_a^b f(x)\, \mathrm{d}x \;.$$

[1]Beware that such class does not exhaust the class of integrable functions.

- The definite integral is additive over adjacent intervals: if $a \le b \le c \in \mathbb{R}$ and $f\colon [a,c] \to \mathbb{R}$ is integrable, then the restrictions $f|_{[a,b]}\colon [a,b] \to \mathbb{R}$ and $f|_{[b,c]}\colon [b,c] \to \mathbb{R}$ are integrable and
$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x\,.$$
As a matter of fact, additivity as in the last displayed equation holds true irrespective of the relative order of a, b and c.
- The definite integral is monotone: if $f, g\colon [a,b] \to \mathbb{R}$ are integrable functions and $f \le g$, that is, $f(x) \le g(x)$ for all $x \in [a,b]$, then
$$\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x\,.$$

C.2. Anti-derivatives and the indefinite integral

Definition C.2.1 (Anti-derivative). Let $f\colon (a,b) \to \mathbb{R}$ be a function, where $a < b \in \mathbb{R}$. An **anti-derivative** of f over (a,b) is a function $F\colon (a,b) \to \mathbb{R}$ which is differentiable on (a,b) and verifies
$$F'(x) = f(x) \quad \text{for every } x \in (a,b).$$

Once an anti-derivative of f over (a,b) is given, it is straightforward to determine all of them.

Proposition C.2.2. *Let $f\colon (a,b) \to \mathbb{R}$ be a function, where $a < b \in \mathbb{R}$, and let $F\colon (a,b) \to \mathbb{R}$ be an anti-derivative of f over (a,b).*

(1) If $C \in \mathbb{R}$, then the function $F + C\colon (a,b) \to \mathbb{R}$, defined as $(F+C)(x) = F(x) + C$ for every $x \in (a,b)$, is an anti-derivative of f over (a,b).
(2) If G is an anti-derivative of f over (a,b), then there exists $C \in \mathbb{R}$ such that $G = F + C$.

The proposition is a consequence of the Mean Value Theorem (Theorem B.3.1) in the theory of differentiation. It leads naturally to the following notion.

Definition C.2.3 (Indefinite integral). Given a function $f\colon (a,b) \to \mathbb{R}$, where $a < b \in \mathbb{R}$, we define the **indefinite integral** of f as the set of all anti-derivatives of f over (a,b):
$$\int f(x)\,\mathrm{d}x = \big\{G\colon (a,b) \to \mathbb{R} : G \text{ is an anti-derivative of } f \text{ over } (a,b)\big\}\,. \tag{C.2.1}$$

The set above is empty whenever f does not admit any anti-derivative. In case there exists an anti-derivative F of f over (a,b) (which, as we shall presently see with the fundamental theorem of calculus, is always the case when f is continuous on (a,b)), then the set in (C.2.1) admits the equivalent description
$$\int f(x)\,\mathrm{d}x = \big\{G\colon (a,b) \to \mathbb{R} : \text{there is } C \in \mathbb{R} \text{ s.t. } G(x) = F(x) + C \text{ for every } x \in (a,b)\big\}$$
or, written concisely,
$$\int f(x)\,\mathrm{d}x = F(x) + C\,, \quad C \in \mathbb{R}.$$

C.3. The fundamental theorem of calculus

What arguably constitutes the crowning achievement of the the Calculus I course is the **Fundamental Theorem of Calculus** (**FTC**), which establishes a close connection between the two operations of differentiation and integration.

THEOREM C.3.1 (Fundamental Theorem of Calculus). *Let* $f\colon [a,b] \to \mathbb{R}$ *be a continuously differentiable function, where* $a < b \in \mathbb{R}$. *Then*

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)\,.$$

REMARK C.3.2. Recall that a function $f\colon [a,b] \to \mathbb{R}$ is said to be continuously differentiable if it is continuous and differentiable on $[a,b]$ (thus including the endpoints) and if the derivative $f'\colon [a,b] \to \mathbb{R}$ is continuous.

The power of the FTC resides in that it allows to compute definite integrals via simple evaluation of an anti-derivative of the integrand at the end points.

A second form of the FTC, following directly from the previous one, is given below.

COROLLARY C.3.3 (Fundamental Theorem of Calculus, second form). *Let* $f\colon (c,d) \to \mathbb{R}$ *be a continuous function, where* $c < d \in \mathbb{R}$. *Given* $a \in (c,d)$, *define a function* $F\colon (c,d) \to \mathbb{R}$ *by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t\,, \quad x \in (c,d).$$

Then $F'(x) = f(x)$ *for every* $x \in (c,d)$.

In a compact form, the theorem states that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)\,.$$