The L^q-spectrum of dynamically driven self-similar measures in arbitrary dimension

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One World Fractals

September 20, 2023

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For every integer $m \geq 1$, let $\mathcal{D}_m = \{2^{-m}(k + [0,1)^d) : k \in \mathbb{Z}^d\}$.

The probability vector $(\mu(Q))_{Q \in \mathcal{D}_m}$ describes how the mass of μ is distributed among cubes of the m-th generation: its *moments* $\sum_{Q \in \mathcal{D}_m} \mu(Q)^q$, $q \in \mathbb{R}_{>1}$, provide an indication as to how close the vector is to being **uniform** ($\mu(Q) = \mu(Q')$ for all Q, Q') or **trivial** ($\mu(Q) = 1$ for some Q).

L^q-spectrum and L^q-dimension

Specifically, if μ is supported inside $[0,1]^d$,

$$2^{-md(q-1)} \leq \sum_{Q \in \mathcal{D}_m} \mu(Q)^q \leq 1 \ ,$$

with equality only occurring in the previous two extreme cases.

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This motivates the following definitions: the **L^q-spectrum** of μ is the function $\tau_\mu \colon \mathbb{R}_{>1} \to \mathbb{R}_{\geq 0}$ defined as

$$\tau_\mu(q) = \liminf_{m \to \infty} -\frac{\log \sum_{Q \in \mathcal{D}_m} \mu(Q)^q}{m} \, , \quad q > 1.$$

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$$\dim_\mu(q) = \frac{\tau_\mu(q)}{q-1}\,, \quad q > 1.$$

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Let $\Phi = (f_i)_{i\in I}$ be a homogeneous self-similar iterated function system (IFS) on $\mathbb{R}^d$: I is a finite set,

$$f_i(x) = \lambda h(x) + a_i \ , \quad x \in \mathbb{R}^d$$

for some $\lambda \in (0,1)$, $h \in \mathrm{SO}_d(\mathbb{R})$, $a_i \in \mathbb{R}^d$ for $1 \le i \le d$.

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Given a probability vector $p = (p_i)_{i \in I}$, let μ be the self-similar measure determined by Φ and p, namely the unique Borel probability measure on $\mathbb{R}^d$ satisfying

$$\mu = \sum_{i \in I} p_i \ f_i \mu \ ,$$

where $f_i \mu$ indicates the pushforward of μ by f_i.

Self-similar measures as infinite convolutions

The measure μ can be alternatively described as the law of the **random infinite sum**

$$\sum_{n\geq 0} \lambda^n h^n(Z_n) ,$$

where $(Z_n)_{n\geq 0}$ is a sequence of i.i.d. random variables with law $\Delta_0 = \sum_{i\in I} p_i \, \delta_{a_i}$. We write μ as the **infinite convolution**

$$\mu = *_{n\geq 0} \, S_{\lambda^n} h^n \Delta_0 ,$$

where $S_{\lambda^n}(x) = \lambda^n x$, $x \in \mathbb{R}^d$, $n \geq 0$.

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where $S_{\lambda^n}(x) = \lambda^n x$, $x \in \mathbb{R}^d$, $n \geq 0$.
The appearance of the iterates $h^n \Delta_0$ suggest the introduction of a **dynamical framework** driving the factors of the infinite convolution product.

A general framework: dynamically driven s.-s. measures

We consider the following general setting: a **uniquely ergodic system** is a triple $(X, \mathbf{T}, \mathbf{P})$ where

- X is a compact metrizable topological space,
- $\mathbf{T} \colon X \to X$ is a continuous transformation,
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A **pleasant model** in $\mathbb{R}^d$ is a quintuple $\mathcal{X} = (\mathsf{X}, \mathbf{T}, \mathbf{P}, \Delta, \lambda)$ where $(\mathsf{X}, \mathbf{T}, \mathbf{P})$ is a uniquely ergodic system, $\lambda \in (0, 1)$ and

$$\Delta\colon \mathsf{X} \to \mathcal{A} = \{\text{finitely supported Borel probability measures in } \mathbb{R}^d\}$$

is a map with the following properties:

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- Δ is measurable, and continuous $\mathbf{P}$-almost everywhere, where $\mathcal{A}$ is endowed with the final topology for the maps $\mathbf{\Delta}_k \times (\mathbb{R}^d)^k \to \mathcal{A},\ ((q_i)_{1\leq i\leq k}, (b_i)_{1\leq i\leq k}) \mapsto \sum_{i=1}^k q_i\ \delta_{b_i},\ k \geq 1$, with $\mathbf{\Delta}_k = \{(q_i)_{1\leq i\leq k} \in (\mathbb{R}_{\geq 0})^k : \sum_i q_i = 1\}$

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- there is an integer $M \geq 1$ and a bounded set $K \subset \mathbb{R}^d$ such that, for all $\mathrm{x} \in \mathrm{X}$, the support of $\Delta(\mathrm{x})$ is contained in K and consists of at most M points.

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A pleasant model $\mathcal{X}$ generates a collection $(\mu_{\mathrm{x}})_{\mathrm{x}\in\mathrm{X}}$ of **dynamically driven self-similar measures**, defined as

$$\mu_{\mathrm{x}} = *_{n\geq 0}\, S_{\lambda^n}\Delta(\mathbf{T}^n\mathrm{x})\ , \quad \mathrm{x} \in \mathrm{X}.$$

L^q-spectrum and L^q-dimensions of a model

They satisfy the **dynamical self-similarity relation**

$$\mu_x = \mu_{x,n} * S_{\lambda^n} \mu_{\mathbf{T}^n x} , \quad x \in X, \; n \geq 1,$$

where the $\mu_{x,n} = *_{i=0}^{n-1} S_{\lambda^i} \Delta(\mathbf{T}^i x)$ are the **level-n approximations**.

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Given a pleasant model $\mathcal{X}$, we define its **L^q-spectrum** and **L^q-dimension**, for every $q > 1$, as

$$T_{\mathcal{X}}(q) = \liminf_{m \to \infty} -\frac{1}{m} \int_X \log \sum_{Q \in \mathcal{D}_m} \mu_x(Q)^q \, d\mathbf{P}(x) \ ,$$

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As a consequence of **Kingman's subadditive ergodic theorem**, the previous are actual limits and, for $\mathbf{P}$-almost every $x \in X$,

$$\tau_{\mu_x}(q) = T_{\mathcal{X}}(q) \ , \quad \dim_{\mu_x}(q) = D_{\mathcal{X}}(q) \quad \text{for every } q > 1.$$

Combinatorial upper bound for the L^q-dimension

A **combinatorial**, general **upper bound** for $D_{\mathcal{X}}(q)$ is readily given.

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so that

$$\tau_{\mu_x}(q) = \liminf_{m \to \infty} -\frac{\log \left\| \mu_x^{(m)} \right\|_q^q}{m}$$

where $\left\| \sum_{j \in J} p_j \ \delta_j \right\|_q^q = \sum_{j \in J} p_j^q$.

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Set also $m(n) = \inf\{m \geq 1 : 2^{-m} \leq \lambda^n\}$. For every $\mathrm{x} \in \mathrm{X}$,

$$\dim_{\mu_{\mathrm{x}}}(q) = \lim_{n \to \infty} -\frac{\log \left\| \mu_{\mathrm{x},n}^{(m(n))} \right\|_q^q}{(q-1)m(n)} \leq \lim_{n \to \infty} -\frac{\log \left\| \mu_{\mathrm{x},n} \right\|_q^q}{(q-1)m(n)}$$

$$\lim_{n \to \infty} -\frac{\log \prod_{i=0}^{n-1} \left\| \Delta(\mathbf{T}^i \mathrm{x}) \right\|_q^q}{(q-1)m(n)} \leq \frac{\int_{\mathrm{X}} \log \left\| \Delta(\mathrm{y}) \right\|_q^q \, \mathrm{d}\mathbf{P}(\mathrm{y})}{(q-1)\log \lambda} =: D_{\mathcal{X}}^s(q) \ .$$

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Even allowing for the occurrence of such degeneracies is not sufficient. Let μ_1 be a **Bernoulli convolution** on $\mathbb{R}$ of parameter $\lambda > 1/2$ satisfying *exponential separation*, μ_2 the standard **Cantor-Lebesgue** measure on the middle-λ^k Cantor set, where $\lambda^k < 1/2$, and let $\mu = \mu_1 \times \mu_2$. Then

$$\begin{aligned} \dim_\mu(q) = \dim_{\mu_1}(q) + \dim_{\mu_2}(q) = 1 + \dim_{\mu_2}(q) < \min\{\dim^s_\mu(q), 2\} \\ = \min\{\dim^s_{\mu_1}(q) + \dim^s_{\mu_2}(q), 2\} \end{aligned}.$$

Forbidding overlaps and saturation on lines

In the latter example, the measure μ is **saturated** on translates of the line $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$, where **excess dimension** is accumulated.

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A pleasant model X satisfies **exponential separation** if, for **P**-almost every $x \in X$, there is $c > 0$ and a subsequence $(n_j)_{j\geq 1}$ such that the atoms of $\mu_{x,n_j} = *_{i=0}^{n_j-1} S_{\lambda^i} \Delta(\mathbf{T}^i x)$ are *distinct* and c^{n_j}-separated.

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For any $\pi \in \mathbb{G}(d, d-1)$, the Grassmannian of $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$, let $\pi\mathcal{X} = (X, \mathbf{T}, \mathbf{P}, \pi\Delta, \lambda)$. We say that $\mathcal{X}$ is ***q*-unsaturated on lines** for some $q > 1$ if

$$D_{\pi\mathcal{X}}(q) > D_{\mathcal{X}}(q) - 1 \quad \text{for every } \pi \in \mathbb{G}(d, d-1) \, .$$

Main result: a formula for the L^q-dimension

Theorem (C.-Shmerkin, 2023)

Let $\mathcal{X}$ be a pleasant model in $\mathbb{R}^d$, generating $(\mu_{\mathsf{x}})_{\mathsf{x}\in\mathsf{X}}$. Assume that X *satisfies exponential separation and is q-unsaturated on lines for some $q > 1$. Then*

$$\lim_{m\to\infty} -\frac{1}{(q-1)m} \log \left\| \mu_{\mathsf{x}}^{(m)} \right\|_q^q = \frac{\int_{\mathsf{X}} \log \|\Delta\|_q^q \, \mathrm{d}\mathbf{P}}{(q-1)\log\lambda} \tag{1}$$

uniformly in $\mathsf{x}\in\mathsf{X}$*. In particular, the limit in the definition of* $\dim_{\mu_{\mathsf{x}}}(q)$ *exists and equals the right-hand side of* (1) *for* **every** $\mathsf{x}\in\mathsf{X}$.

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This encompasses the case $d = 1$ (Shmerkin, 2019), where only exponential separation is needed: q-unsaturation amounts to $D_{\mathcal{X}}(q) < 1$, in the absence of which $\dim_{\mu_{\mathrm{x}}}(q) = 1$ holds tautologically.

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$$\dim_{\pi\mu}(q) > \dim_\mu(q) - 1 \quad \text{for all } \pi \in \mathbb{G}(d, d-1) \, .$$

Corollary

Let μ be a homogeneous self-similar measure in $\mathbb{R}^d$ generated by an IFS Φ and a probability vector p. Assume Φ satisfies exponential separation and μ is q-unsaturated on lines for some $q > 1$. Then

$$\dim_\mu(q) = \dim_\mu^s(q) = \frac{\log \|p\|_q^q}{(q-1)\log \lambda} \, .$$

Deduction from the main theorem

Write $\Phi = (f_i)_{i\in I}$, $f_i(x) = \lambda h(x) + a_i$, $x \in \mathbb{R}^d$. It suffices to apply the theorem to $(\mathrm{X}, \mathbf{T}, \mathbf{P}, \Delta, \lambda)$ given as follows:

- X is the closure in $\mathrm{SO}_d(\mathbb{R})$ of the cyclic subgroup generated by h;
- $\mathbf{T}\colon \mathrm{X} \to \mathrm{X}$ is translation by h;
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Unique ergodicity is a general feature of **translations on compact abelian groups** by elements generating a **dense cyclic** subgroup.

Super-critical regime and the planar case

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Remark: in the **super-critical regime** $\dim_\mu^s(q) > d$, it is expected under favourable circumstances that $\dim_\mu(q) = d$; however, this automatically **rules out q-unsaturation**. The issue is taken care of by **thinning** the driving model $\mathcal{X}$ into a new pleasant model $\mathcal{X}^{\text{th}}$ satisfying $D_{\mathcal{X}^{\text{th}}}(q) \leq D_{\mathcal{X}}(q)$ and with $d > D^s_{\mathcal{X}^{\text{th}}}(q)$ arbitrarily close to d. Applying the theorem to $\mathcal{X}^{\text{th}}$, for which q-unsaturation has a chance of holding, yields $\dim_\mu(q) = D_{\mathcal{X}}(q) = d$.

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Corollary

Let Φ be a homogeneous self-similar IFS in $\mathbb{R}^2$ as above. Assume h is an irrational rotation and Φ satisfies exponential separation. Then, for every vector p and every $q > 1$, the s.-s. m. μ generated by (Φ, p) satisfies

$$\dim_\mu(q) = \min\left\{\dim_\mu^s(q), 2\right\}.$$

The proof: projecting and inducting on the dimension

Remark: when $h = \mathrm{id}_{\mathbb{R}^2}$, the result **fails**, as in the counterexample of a product measure $\mu = \mu_1 \times \mu_2$ discussed above.

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The proof in a nutshell: when $\dim_\mu^s(q) < 2$, q-unsaturation holds for the following reason. For any $\pi \in \mathbb{G}(2,1)$, the one-dimensional model $\pi\mathcal{X}$ satisfies exponential separation, so that $D_{\pi\mathcal{X}}(q) = \min\{D^s_{\pi\mathcal{X}}(q), 1\}$. Irrationality of h implies $D^s_{\pi\mathcal{X}}(q) = D^s_{\mathcal{X}}(q)$, whence

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Such strategy of **projecting** and **inducting on the dimension** can be implemented in higher dimensions.

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$$= \frac{\int_X \log \|\Delta(y)\|_q^q \, d\mathbf{P}(y)}{(q-1)\log \lambda},$$

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holding for $\mathbf{P}$-almost every $x \in X$. Fix such an x for which exponential separation holds; then, for some sufficiently large integer $R \geq 1$,

$$\left\| \mu_{x,n}^{(Rm(n))} \right\|_q^q = \|\mu_{x,n}\|_q^q = \prod_{i=0}^{n-1} \left\| \Delta(\mathbf{T}^i x) \right\|_q^q .$$

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It remains to show that $D_{\mathcal{X}}(q) = \lim_{n\to\infty} -\frac{\log \left\|\mu_{x,n}^{(Rm(n))}\right\|_q^q}{(q-1)m(n)}$.

L^q-norm at finer scales via L^q-flattening

Inductively on $R \geq 1$: by dynamical self-similarity,

$$\left\| \mu_{\mathrm{x}}^{((R+1)m(n))} \right\|_q^q = \Theta_q(1) \left\| \mu_{\mathrm{x},n}^{((R+1)m(n))} * (S_{\lambda^n} \mu_{\mathbf{T}^n \mathrm{x}})^{((R+1)m(n))} \right\|_q^q ,$$

where now

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$$\left\| \mu_{\mathrm{x}}^{((R+1)m(n))} \right\|_q^q = \Omega_{\lambda,q}(1) 2^{-(T_{\mathcal{X}}(q)+\delta)(R+1)m(n)} , \quad \delta > 0 \text{ fixed close to } 0.$$

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Proposition (L^q-smoothening)

Suppose $\mathcal{X}$ is q-unsaturated and $T_{\mathcal{X}}(q) < d(q-1)$. For every $\sigma > 0$, there exists $\varepsilon = \varepsilon(q,\sigma)$ such that, for every sufficiently large $m \in \mathbb{N}$, every 2^{-m}-measure satisfying $\|\nu\|_q^q \leq 2^{-\sigma m}$ and every $\mathrm{x} \in \mathrm{X}$,

$$\left\|\nu * \mu_{\mathrm{x}}^{(m)}\right\|_q^q \leq 2^{-(T_{\mathcal{X}}(q)+\varepsilon)m} .$$

Inverse theorem for L^q-norms of convolutions

Theorem (Shmerkin, 2023)

Let $q > 1$, $\delta > 0$. For every sufficiently large $D \in \mathbb{N}$ there is $\varepsilon > 0$ such that the following holds for every $\ell \geq \ell_0 \in \mathbb{N}$. Let $m = \ell D$, μ, ν 2^{-m} measures in $[0,1)^d$ satisfying

$$\|\mu * \nu\|_q \geq 2^{-\varepsilon m} \|\mu\|_q \ .$$

Then there exist $A \subset \operatorname{supp} \mu$, $B \subset \operatorname{supp} \nu$ and sequences $(R'_s)_{s\in[\ell]}$, $(R''_s)_{s\in[\ell]}$ ($[\ell] = \{0, \ldots, \ell - 1\}$) of natural numbers such that

- $\|\mu|_A\|_q \geq 2^{-\delta m} \|\mu\|_q$ *and* $\mu(x) \leq 2\mu(y)$ *for all* $x, y \in A$;
- *for all* $s \in [\ell]$ *and* $Q \in \mathcal{D}_{sD}(A) = \{Q \in \mathcal{D}_{sD} : Q \cap A \neq \emptyset\}$, $\mathcal{N}_{(s+1)D}(A \cap Q) = |\mathcal{D}_{(s+1)D}(A \cap Q)| = R'_s$;
- $\nu(B) \geq 2^{-\delta m}$ *and* $\nu(x) \leq 2\nu(y)$ *for all* $x, y \in B$;
- *for all* $s \in [\ell]$ *and* $Q \in \mathcal{D}_{sD}(B)$, $\mathcal{N}_{(s+1)D}(B \cap Q) = R''_s$.

Moreover, for every $s \in [\ell]$, either $R''_s = 1$ or, for every $Q \in \mathcal{D}_{sD}(A)$, there is $\pi_Q \in \mathbb{G}(d, d-1)$ such that $\mathcal{N}_{(s+1)D}(A \cap Q) \geq 2^{(1-\delta)D} \mathcal{N}_{(s+1)D}(\pi_Q(A \cap Q))$.