ON THE SUBSTITUTION RULE

Week 3 small class

The following is a reminder from the last lecture, which dealt with the substitution rule for integrals. If F, f are functions such that

$$\int f(x)\,\mathrm{d}x = F(x) + C\ ,$$

then, for every third function φ,

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = F(\varphi(x)) + C$$

In concrete computations, the rule amounts to symbolically replacing $\varphi(x)$ by a new variable u and $\varphi'(x)\,\mathrm{d}x$ by $\mathrm{d}u$, so as to obtain

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u\ , \quad \text{with } u = \varphi(x)\ .$$

In the upcoming three problems, we shall apply the substitution rule in combination with a new technique, called *partial fraction decomposition*.

Problem A

(1) Which would you rather integrate?

$$\text{A)}\quad \int\left(\frac{1}{x-1} - \frac{1}{2x+1}\right)\mathrm{d}x \qquad\qquad \text{B)}\quad \int\left(\frac{x+2}{2x^2-x-1}\right)\mathrm{d}x$$

Solution. The first integral looks easier, but more importantly it is susceptible to a technique we learned about in the large class. First, we separate the two summands in the integral using the property of integrals discussed in the first small class:

$$\int\left(\frac{1}{x-1} - \frac{1}{2x+1}\right)\mathrm{d}x = \int \frac{1}{x-1}\,\mathrm{d}x - \int \frac{1}{2x+1}\,\mathrm{d}x\ . \tag{1}$$

To each summand, we now apply the substitution rule. For the first one, we set $u = x-1$, so that $\mathrm{d}u = (x-1)'\,\mathrm{d}x = \mathrm{d}x$, and thus

$$\int \frac{1}{x-1}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u = \log|u| + C = \log|x-1| + C\ . \tag{2}$$

Similarly, for the second one we set $u = 2x+1$, from which $\mathrm{d}u = (2x+1)'\,\mathrm{d}x = 2\,\mathrm{d}x$, that is, $\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$. Hence

$$\int \frac{1}{2x+1}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\log|u| + C = \frac{1}{2}\log|2x+1| + C\ . \tag{3}$$

Combining (1), (2) and (3), we obtain

$$\int\left(\frac{1}{x-1} - \frac{1}{2x+1}\right)\mathrm{d}x = \log|x-1| - \frac{1}{2}\log|2x+1| + C\ .$$

(2) What happens if we find a common denominator and add the fractions in integral A)?

Solution. We find that the integrals in A) and B) are the same. Indeed,

$$\frac{1}{x-1} - \frac{1}{2x+1} = \frac{2x+1-(x-1)}{(x-1)(2x+1)} = \frac{x+2}{2x^2-x-1}\ .$$

[**Takeaway**] The integral in B) would be approachable but we need a way to "undo" finding a common denominator. This method is called the method of *partial fractions*.

Problem B

(1) Consider $\int \frac{7x+13}{(2x+5)(x-2)}\,\mathrm{d}x$. Start by supposing our function can be written in the following way:
$$\frac{7x+13}{(2x+5)(x-2)} = \frac{A}{2x+5} + \frac{B}{x-2}\,.$$
Find A and B.

Solution. We have
$$\frac{A}{2x+5} + \frac{B}{x-2} = \frac{A(x-2)+B(2x+5)}{(2x+5)(x-2)} = \frac{(A+2B)x+5B-2A}{(2x+5)(x-2)}\,.$$
As we want the latter to be equal to
$$\frac{7x+13}{(2x+5)(x-2)}$$
for every $x \in \mathbb{R}$, we must equate the coefficients corresponding to monomials of the same degree, whence we need to solve the linear system
$$\begin{cases} A+2B=7 \\ 5B-2A=13 \end{cases}.$$
The first equation gives $A = 7 - 2B$, which we can plug into the second equation, thereby getting
$$5B - 2(7-2B) = 13\,, \text{ that is, } 9B = 27\,,$$
which gives $B = 3$, whence $A = 7 - 2 \cdot 3 = 1$.

Alternatively, we may observe that the equality
$$A(x-2) + B(2x+5) = 7x + 13$$
must be valid for *every* $x \in \mathbb{R}$; therefore, we may judiciously choose values of x which simplify our search for A and B. For instance, taking $x = 2$ makes the summand containing the factor A vanish, and yields
$$A \cdot 0 + B(4+5) = 7 \cdot 2 + 13 = 27\,,$$
which immediately gives $B = 3$. Likewise, taking $x = -\frac{5}{2}$ makes the term containing the factor B vanish:
$$A\left(-\frac{5}{2} - 2\right) + B \cdot 0 = -7 \cdot \frac{5}{2} + 13\,, \text{ that is, } -\frac{9}{2}A = -\frac{9}{2}\,,$$
and $A = 1$ follows.

[**Takeaway**] When doing partial fractions, selecting convenient values of x can simplify your algebra.

(2) Compute the integral.

Solution. From the previous point, and the well-known property of the integral of a sum of two functions, we have
$$\int \frac{7x+13}{(2x+5)(x-2)}\,\mathrm{d}x = \int \frac{1}{2x+5}\,\mathrm{d}x + 3\int \frac{1}{x-2}\,\mathrm{d}x\,.$$
Substituting $u = 2x+5$, $\mathrm{d}u = 2\,\mathrm{d}x$ in the first integral, we get
$$\int \frac{1}{2x+5}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\log|u| + C = \frac{1}{2}\log|2x+5| + C\,;$$

similarly, for the second integral, we substitute $u = x - 2$, $\mathrm{d}u = \mathrm{d}x$ and obtain

$$\int \frac{1}{x-2}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u = \log|u| + C = \log|x-2| + C\,.$$

Putting everything together, we conclude that

$$\int \frac{7x+13}{(2x+5)(x-2)}\,\mathrm{d}x = \frac{1}{2}\log|2x+5| + 3\log|x-2| + C\,.$$

[**Takeaway**] Write the big fraction as two simpler fractions and solve for the numerator.

Problem C

Compute $\int \sec x\ \mathrm{d}x$, by first transforming it into $\int \frac{\cos x}{\cos^2 x}$ and then transforming the latter into $\int \frac{1}{1-u^2}\ \mathrm{d}u$.

Solution. Recall the definition of the secant function: $\sec x = \frac{1}{\cos x}$ for every $x \in \mathbb{R}$ for which $\cos x \neq 0$. Multiplying numerator and denominator by the non-zero quantity $\cos x$, we get $\sec x = \frac{\cos x}{\cos^2 x}$, whence in particular

$$\int \sec x\ \mathrm{d}x = \int \frac{\cos x}{\cos^2 x}\ \mathrm{d}x\,.$$

We now make use of the trigonometric identity

$$\cos^2 x + \sin^2 x = 1\,,$$

valid for every $x \in \mathbb{R}$, in order to derive

$$\cos^2 x = 1 - \sin^2 x\,.$$

We are thus left with finding

$$\int \frac{\cos x}{1-\sin^2 x}\ \mathrm{d}x\,.$$

Substituting $u = \sin x$, which yields $\mathrm{d}u = (\sin x)'\ \mathrm{d}x = \cos x\ \mathrm{d}x$, we see that the latter indefinite integral is precisely equal to

$$\int \frac{1}{1-u^2}\ \mathrm{d}u\,,$$

which in turn can be approached via the partial-fraction decomposition technique learned above. The denominator $1-u^2$ factors as $(1-u)(1+u)$, and imposing the condition

$$\frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$$

on $A, B \in \mathbb{R}$ gives, after easy computations, $A = B = 1/2$. It follows, again by substitution, that

$$\int \frac{1}{1-u^2}\ \mathrm{d}u = \frac{1}{2}\int \frac{1}{1+u}\ \mathrm{d}u + \frac{1}{2}\int \frac{1}{1-u}\ \mathrm{d}u = \frac{1}{2}\log|1+u| - \frac{1}{2}\log|1-u| + C\,.$$

Recalling that $u = \sin x$, we conclude that

$$\int \sec x\ \mathrm{d}x = \frac{1}{2}\log|\sin x + 1| - \frac{1}{2}\log|\sin x - 1| + C\,.$$

[**Takeaway**] Sometimes we combine partial fractions with other techniques.